Contents

Alternative and Indefinitely Repeated Investments: Species Choice and Harvest Age in Forestry[1]

Skander Ben Abdallah
Université du Québec à Montréal
ben_abdallah.skander@uqam.ca

Pierre Lasserre
Université du Québec à Montréal,
CIREQ, and CIRANO
lasserre.pierre@uqam.ca

In progress, July 2012

---

[1] We are grateful to SSHRC, FQRSC, and the Sustainable Forest Management Network for financial support.

Abstract

The tree cutting problem of forest management is an archetypal investment problem; it involves time, uncertainty, and irreversible actions with consequences in the future. The exercise of the option to cut a tree opens the option of planting a new one or of using the land for alternative purposes. We enrich the tree cutting problem by considering the planting decision too with no cost associated with harvesting or planting. Two tree species are available; their growth is deterministic but their timber unit price is stochastic. In the case of a single rotation, known as the Wicksellian tree cutting problem, the forest manager should plant one species immediately if its price is sufficiently high relative to the price of the other species. However, if prices are close to each other, the manager should wait in order to avoid the mistake of planting the wrong species. Waiting should last until the prices are sufficiently far apart to make the probability of a future price reversal acceptably low. In contrast, when the number of rotations is arbitrarily high, as in Faustman's problem, waiting before planting the new tree, whatever its species, is never optimal once a harvest has taken place. However, the optimal cutting age depends on the relative timber price. We show that the optimum cutting age increases when the relative price approaches a threshold value signalling the necessity to switch to the alternative species. This is because the decision maker would rather wait than plant the wrong species; letting existing trees grow older is a way to postpone the choice. We also show that the land value increases with the uncertainty of timber prices. The stand value is similar to the value of an American option with a free boundary and an expiry date equals to infinity but with endogenous payoff.

key words: Real options; stochastic prices; American options; free boundary; penalty method; forestry; Faustman; alternative species; rotation.

J.E.L. classification: C61; D81; G11; G13; Q23.

Résumé

La décision de couper un arbre en gestion forestière est l'archétype du problème d'investissement; il implique le temps, l'incertitude ainsi que l'irréversibilité d'actions ayant des conséquences dans le futur. L'exercice de l'option de couper un arbre donne naissance à l'option de planter un nouvel arbre ou bien d'utiliser la terre à d'autres fins. On enrichit ce problème en considérant aussi la décision de planter. Deux essences d'arbres sont disponibles; leurs fonctions de croissance sont déterministes mais les prix de leurs bois sont stochastiques. Lorsqu'une seule rotation est envisagée, il s'agit du problème dit de Wicksell; le gestionnaire forestier plante immédiatement une essence si le prix de son bois est suffisamment élevé relativement au prix de l'autre essence. Cependant, si les deux prix sont proches, le gestionnaire doit attendre dans le but de ne pas commettre l'erreur de choisir la mauvaise essence. L'attente doit se prolonger jusqu'à ce que les prix se distinguent suffisamment de telle sorte que la probabilité que l'ordre des prix s'inverse dans le futur devienne suffisamment faible. En revanche, quand le nombre de rotations est illimité, on se réfère au problème de Faustman; il n'est jamais optimal d'attendre avant de planter une essence une fois qu'une récolte a eu lieu. Cependant, l'âge des arbres à la récolte dépend du prix relatif de leur bois. On montre que l'âge optimal à la récolte croît quand le prix relatif est suffisamment proche d'un seuil signalant la nécessité de planter l'essence alternative. Ceci est expliqué par le désir du gestionnaire d'attendre plutôt que de planter la mauvaise essence; laisser croître plus longtemps les arbres est une manière de reporter la prise de décision. La valeur d'un terrain planté est similaire à la valeur d'une option américaine avec une frontière libre et une échéance infinie d'expiration mais un paiement endogène.

mots clés: Options réelles; prix stochastiques; options américaines; frontière libre; méthode de pénalité; essences alternatives; rotation.
Qualification JEL: C61, D81, G11, G13, Q23.

1. INTRODUCTION

Forest management is an archetypal investment problem; it involves time; it involves uncertainty; it involves irreversible actions with consequences in the future. It also exemplifies investments that open up new options: cutting a tree opens the option of planting a new one or using the land differently. Faustman (1849) gave forestry economics its foundations by addressing the question: at what age should a stand of even-aged trees be harvested? He did so under the assumption of constant timber prices by comparing the net marginal benefits from letting timber grow further, to the opportunity cost of existing trees plus the opportunity cost of the land, itself a function of timber management decisions.

Faustman's original problem has been refined and generalized in many ways. In this paper we focus on the availability of alternative species to replace the trees being harvested. We show how this choice should be made and timed, and how it affects the harvesting decision. While species choice is clearly important for the forester, that question has much wider relevance and may be viewed as illustrating a general investment problem. The question of choosing a harvest age is akin to that of deciding when an equipment should be retired; the species choice is similar to the choice of alternative technologies for replacement. Thus we introduce alternative projects in situations where decisions open up new options in a process that repeats itself indefinitely, and must be reevaluated at each instant. This most common practical situation has not been investigated theoretically before, although both the real option literature and the forestry literature have gone some way toward solving that problem.

The real option approach has been widely applied in natural resources exploitation and management. In the real option framework, a typical investment involves an optimum stopping rule, the choice of the date at which the decision maker considers that conditions have become favorable enough to justify committing resources irreversibly. A typical result is that more uncertainty postpones investment although it increases project value.

Applying this approach, a number of studies examine the optimal harvesting age in forestry under stochastic timber price. The focus has mainly been on optimal harvesting when observing stochastic timber prices as in Brazee and Mendelsohn (1988) and Insley (2002). Willassen (1998) dismisses the optimal stopping methodology and uses impulse control. These studies show that when timber prices or stand value follow a Brownian motion and in absence of management costs, the solution is similar to the deterministic case. However, when management costs are considered, the expected value of the stand is increased by considering a reservation price policy that exploits stochastic variations in prices. Uncertainty provides an incentive to postpone harvest and delay management costs until observing future prices and schedule harvests accordingly. Platinga (1998) shows that the option value to delay harvest when timber prices are stationary stems from the level of the stumpage price with respect to the long-term mean and the stand value with respect to fixed management cost. Over time, applications have been extended to include more and more problems, such as differentiated timber prices (Forboseh et al., 1996), uneven-aged management (Haight, 1990), multi-species stands under changing growth conditions caused by climate change (Jacobsen and Thorsen, 2003).

The recent real option literature has begun to treat situations where an action of the decision maker simultanously involves a choice between alternative opportunities and choosing the timing of an investment. These alternative opportunities may differ with respect to investment costs and output amount as in Decamps et al (2006) or benefit trajectories as in Kassar and Lasserre (2004). An interesting result in such situations is that a new reason for postponing action arises. When the alternatives are too close to each other and uncertain, the decision maker may choose to wait in order to avoid choosing an alternative that might prove to be less desirable than another candidate in the future. This inaction may be optimal although each project, taken in isolation, would satisfy the requirements for immediate investment under conditions of irreversibility and uncertainty. Although the forestry literature has considered choices between alternative options (Reed, 1993; Conrad, 1997; Abildtrup and Strange, 1999) such as the investment problem of stand establisment, no attention has been devoted to the hysteresis possibly induced in such circumstances. Furthermore, when simultanous alternatives were considered, the future stand value was treated as exogenous, independent of the current choice. For instance, Thorsen (1999*a*) analyses the choice of tree species for afforestation as a real option problem, and Thorsen and Malchow-Møller (2003*a*) extend it to a two-option problem with two mutually exclusive options (two tree species), where exercising one option implies losing the other. Jacobsen (2007) goes one step further: upon harvest, the current stand (of spruce) may be allowed to regenerate naturally and costlessly, or may be replaced with oaks at some cost. However, it is assumed that at some finite time horizon oaks will be definitely planted so that the problem eventually simplifies to the indefinitely repeated single-species problem of Faustman, with stochastic price. Finally, it is not certain whether the age at which harvest is optimal is higher or lower than Faustman's or Wicksell's rotation.

To our knowledge our paper is the first one involving an indefinitely repeated choice between options. Given an existing stand composed of one of two possible species, the decision maker chooses the harvest age. Then, she decides which species or which proportion of species should be used to repopulate the land, where timber prices evolve stochastically, possibly with some correlation. Then again, and forever, harvests times and species choices must be decided optimally. To draw again on the analogy with equipement retirement and replacement, equipment may be retired at any age and replaced with any of two alternative technologies where the efficiency of each technology evolves stochastically as suppliers improve their products. It is not certain that one type will dominate the other for long, let alone forever.

The general setting and assumptions are introduced in Section 2. After harvesting, the land may be repopulated with any combination of two tree species; the new crop may be established right after the harvest or at any later time. Each species is characterized by a different, stochastic, timber price process and by a distinct, although not stochastic, growth function. In Section 3, we investigate the case of a single rotation, also known as Wicksell's tree cutting problem. Stochastic versions of that problem for a single species have been discussed by Willassen (1998). With two species implies one should consider the species choice and its timing, before considering harvest. A form of hysteresis not previously identified in the literature arises: under some conditions about relative species prices, the decision maker waits rather than establishing a crop, despite the fact that each species would be worth planting in isolation. In Section 4 we extend the analysis to multiple rotations. The decision maker must decide at what age the trees of the current stand must be cut; after harvest she must decide whether she should establish a new crop or wait; when she makes such decision, she must choose one, or the other, species. This process is repeated indefinitely. We find that the hysteresis of the Wicksellian problem disappears.

The qualitative properties of the decision rules and value functions are described analytically and presented in a number of propositions.[2] In particular, we show that, under uncertainty, the stand value is similar to the value of an American option with a free boundary and an expiry date equal to infinity but with endogenous payoff. The analysis is completed with a numerical resolution based on the penalty method (Zvan et al 1998) applied simultaneously to the stand value function of each species, and on a Newton iterative process applied to the land value. We show analytically that the optimum harvest age increases when the relative price approaches some threshold value signalling the necessity to switch to the alternative species. This is because the decision maker would rather wait than plant the wrong species; letting existing trees grow older is a way to postpone the choice, allowing the prices to become more informative. Conversely, if the relative price exceeds the switching threshold, the risk of making a mistake by switching diminishes as the relative price moves further away from the threshold, so that the reason to postpone harvest becomes less compelling and the optimum harvest age diminishes. This possibility to postpone harvest explains why the hysteresis observed in the Wicksellian problem, taking the form of forest land being left bare until a decision to plant is made, disappears when harvests can be indefinitely

---

[2] The assumption that prices follow geometric Brownian motions helps obtain some of the analytical results. However the numerical treatment does not depend on that assumption and the qualitative properties of the solution, as well as the intuitive understanding arising from the analytical results still stand to changes in it.

repeated. Under uncertainty, land value increases with the volatility of timber prices. It is continuous and differentiable even around the switching threshold, which renders profitable arbitrage impossible. Finally, Section 5 concludes.

2. GENERAL SETTING AND ASSUMPTIONS

We study the decision by a risk-neutral forest manager to establish one, or any mix, of two alternative tree species P and P' on a plot of bare land. The timber price of species P (respectively P') follows a geometric Brownian motion (GBM) with drift μ (resp. μ') and volatility σ (resp. σ'):

$$dp = \mu p dt + \sigma p dz \tag{1a}$$

$$dp' = \mu' p' dt + \sigma' p' dz \tag{1b}$$

where time indices have been omitted, $dz = \varepsilon\sqrt{dt}$ and $dz' = \varepsilon'\sqrt{dt}$ are the increments of Wiener processes, and ε and ε' are standardized Gaussian white noises whose correlation is ρ. In the rest of the paper variables that depend on time are indexed unless they are considered at the current date defined as $t = 0$, in which case the time index is omitted. The relative price $\theta = \frac{p'}{p}$ is time variable while $\delta = r - \mu > 0$ and $\delta' = r - \mu' > 0$[3] are constant parameters, where r is the discount rate.

Each tree species is characterized by a timber volume growth function with the following properties:

Assumption 1 *There exists $\underline{a} > 0$ and $\underline{a}' > 0$, such that the timber volume functions $V(a)$ and $V'(a)$ are continuous over $[0, +\infty[$, $V(a) = 0$ over $[0, \underline{a}]$, $V'(a) = 0$ over $[0, \underline{a}']$; $V(a)$ and $V'(a)$ are positive, continuous, differentiable and concave over $[\underline{a}, +\infty[$ and $[\underline{a}', +\infty[$ respectively. In addition, $\lim_{a\to+\infty} V_a(a) = 0$ and $\lim_{a\to+\infty} V'_a(a) = 0$.*

Volume growth functions usually have a convex initial part and become concave once the trees have reached some strictly positive age. This implies that it is never optimum to harvest at an arbitrarily low age. Assumption 1 ensures that this stylized property is

---

[3] We assume that $\delta > 0$ and $\delta' > 0$; otherwise it would be optimal to delay the investment forever.

satisfied while avoiding delicate and economically trivial complications associated with the non convexity of the volume functions at low tree ages[4]. Similarly, we assume that operational costs (while trees are growing) and harvesting costs are either nil or accounted for in prices p and p'.

A more consequential assumption is that planting and other initial investment costs are zero. We make that assumption for two reasons. The first one is theoretical. A key element of real options decisions is the irreversible commitment of resources at the time a project is undertaken. When these resources take the form of an irreversible investment cost, and the future returns of the project are uncertain, they create a deferral option or an option to wait that is well understood.[5] This paper focuses on the option to choose, once or an indefinite number of times, between two alternatives; this option is best identified in the absence of sunk costs, an assumption that eliminates the conventional deferral option impact. The second reason for eliminating investment costs is that this assumption allows to go further into the analytical resolution of the problems to describe the impact of the switching option on the investment decision. The numerical method developed to illustrate the solution and to complement the results can be adapted to a model involving a positive planting or investment cost.

3. Choosing between two investment projects: an extension of Wicksell's problem

"The Wicksellian tree cutting problem" refers to the problem of choosing the age at which a stand of even-aged trees will be harvested. One single harvest is considered. The optimal harvesting age is determined by the well-known Wicksellian rule under which the optimal age is chosen in such a way that the marginal value growth of the trees is equal to the opportunity cost of holding them.

We modify the stochastic version of Wicksell's problem in two essential ways. First

---

[4] In the numerical illustrations, we use the same volume growth function for both species; this better isolates the role of the stochastic price processes defined by (1a) and (1b). The volume growth function used is $V(a) = V_\infty \left(1 - e^{-\alpha(a-\underline{a})}\right)$ where $V_\infty = 100$ is the timber volume when the age tends to infinity, $\alpha = 0.01$, and $\underline{a} = 10$ is the minimum age for positive growth.

[5] See the references in the introduction.

we start with bare land and consider the tree planting decision. Second we study a situation where two tree species, not simply one, are available. The reasons why these new features are important are the following. First, since the timber value of each species evolves stochastically, the revenues derived upon harvest depend on the species initially planted and on the price reached at harvesting. The harvest value of one species may overtake the other one, implying that the decision maker may regret the initial choice. Second, the availability of more than one species opens up the possibility of diversification that we show is not optimal in the following proposition.

Proposition 1 *When two tree species may be grown simultaneously on a forest land, it is optimal to specialize into one, or the other, species rather than diversify.*

Proof. See the appendix. ■

Proposition 1 indicates that the stand value at planting time is highest when one single species, rather than a combination of the two species, is established on the plot. Clearly, however, a combination with lower expected value but less risk might be preferable if the decision maker was risk averse. Let $G(p,a)$ $(G'(p',a))$ be the stand value function when species P (resp. species P) is in place, the age of the trees is a, and the current price of the species in place is p (resp. p'):

$$\begin{aligned} G(p,a) &= \max_s E\left[e^{-rs}V(a+s)p_s\right] && \text{(2a)} \\ G'(p',a) &= \max_s E\left[e^{-rs}V'(a+s)p'_s\right] && \text{(2b)} \end{aligned}$$

Consider $G(p,a)$. As $G(p,a)$ is homogenous in p, $G(p,a) = pg(a)$ where $g(a) \equiv G(1,a)$. $G(p,a)$ must satisfy Bellman's equation $EdG = rGda$ which implies, by Itô's lemma, that $g(a)$ satisfies $\delta g(a) - g_a(a) = 0$. Then $g(a) = be^{\delta a}$, where b is a constant to be determined using the value-matching and smooth-pasting conditions as follows. At cutting age a_w and for any price p, $G(p,a_w) = V(a_w)p$, $G_p(p,a_w) = V(a_w)$, and $G_a(p,a_w) = V_a(a_w)p$, implying $g(a_w) = V(a_w)$ and $g_a(a_w) = V_a(a_w)$. Consequently,

$$\begin{aligned} G(p,a) &= be^{\delta a}p && \text{(3a)} \\ b &= V(a_w)e^{-\delta a_w} && \text{(3b)} \\ \frac{V_a(a_w)}{V(a_w)} &= \delta && \text{(3c)} \end{aligned}$$

where a_w is determined implicitly by the last equation as a function of δ independent of price. In particular when the expected value of timber is constant ($\mu = 0$), the optimum harvest age is given by condition $\frac{V_a(a_w)}{V(a_w)} = r$, which is Wicksell's rule in the absence of uncertainty (Willassen, 1998).

To examine the effect of the existence of the second tree species, note similarly that

$$\begin{aligned} G'(p',a) &= b'e^{\delta' a}p' && \text{(4a)} \\ b' &= V'(a'_w)e^{-\delta' a'_w} && \text{(4b)} \\ \frac{V'_a(a'_w)}{V'(a'_w)} &= \delta' && \text{(4c)} \end{aligned}$$

In particular, when the age of the trees is zero, the stand value is either bp or $b'p'$, depending on the species. If the species is yet to be chosen, the stand value is $\max_T Ee^{-rT}\left\{\left[bp_{_T}, b'p'_{_T}\right]\right\}$, where T is the date at which the chosen species will be established. As only one harvest is possible in Wicksell's problem, this is also the value of the bare land:

$$F(p,p') = \max_T Ee^{-rT}\left\{\left[bp_{_T}, b'p'_{_T}\right]\right\}. \tag{5}$$

Since $F(p,p')$ is homogenous in (p,p') it can be written $F(p,p') = pf(\theta)$ where $\theta = \frac{p'}{p}$.

Suppose that the land is currently bear and that it is optimal to wait rather than establish a new crop at the current time. Then there exists a time interval dt during which $F(p,p')$ satisfies Bellman's Equation $EdF = rFdt$. This defines the continuation region in the plan (p,p'). Itô's lemma implies that $f(\theta)$ satisfies:

$$\frac{\overline{\sigma}^2}{2}\theta^2 f_{\theta\theta}(\theta) + \overline{\mu}\theta f_\theta(\theta) - \delta f(\theta) = 0 \tag{6}$$

with $\overline{\sigma}^2 \equiv \sigma^2 - 2\rho\sigma\sigma' + \sigma'^2$ and $\overline{\mu} \equiv \mu' - \mu = \delta - \delta'$.

This differential equation has solutions (*i.e.* the continuation region exists) when its characteristic equation $Q(\beta) = 0$ has real roots, where $Q(\beta) = \frac{\overline{\sigma}^2}{2}\beta(\beta-1) + \overline{\mu}\beta - \delta$. This requires the equation to be quadratic and its determinant $\Delta \equiv (\overline{\mu} - \frac{\overline{\sigma}^2}{2})^2 + 2\overline{\sigma}^2\delta$ to be non negative. The equation is quadratic if and only if either $\sigma > 0$ or $\sigma' > 0$. If this condition does not hold, the species yielding the highest return at harvest is known for any crop establisment time. Since $\mu < r$ and $\mu' < r$, it is more profitable to establish a given species immediately than later. Consequently it is never optimal to wait before choosing a species if $\sigma = \sigma' = 0$. The determinant of the characteristic equation is strictly positive if and only if ($\mu = \mu'$, $\sigma = \sigma'$, $\rho = 1$) or ($\mu = \mu'$, $\sigma = \sigma' = 0$). In such cases, prices follow parallel trajectories, so that the higher price will remain so forever; it is then optimal to invest immediately in the project with the highest price. To rule out these trivial cases where the continuation region does not exist, we make the following assumption:

Assumption 2 *The future price of at least one species is uncertain i.e.* $\sigma > 0$ *or* $\sigma' > 0$ *and* $(\mu, \sigma) \neq (\mu', \sigma')$.

Under Assumption 2, the characteristic equation has two roots $\beta_1 > 1$ and $\beta_2 < 0$. In that case,

$$f(\theta) = b_1\theta^{\beta_1} + b_2\theta^{\beta_2} \tag{7}$$

where b_1 and b_2 are two constants. The option value function is then $F(p, p') = pf(\theta)$ in the continuation region. However, at some levels of θ, it is optimal to invest rather than wait. Precisely,

Proposition 2 *(one rotation) When one of two tree species must be established for a single harvest, the optimal decision is to leave the land bare unless or until the relative timber price is out of an interval* $\left[\theta^*,\ \theta^{'*}\right]$. *The first time that the relative price reaches* $\theta^{'*}$ *from below (resp.* θ^* *from above) species* P' *(resp.* P*) should be established. Once species* P *(resp.* P'*) has been established the stand should be allowed to grow until it reaches its optimal harvest age defined by Formula* (3c) *(resp.* (4c)*), which is independent* *of the relative price. As long as the land is bare, its value is* $pf(\theta)$ *given by* (7)*; once populated with age* a *trees of species* P *(resp.* P'*), the land is worth* $be^{\delta a}p$ *(resp.* $b'e^{\delta' a}p'$*) where* b *is given by* (3*b*) *(resp.* b' *is given by* (4*b*)*).*

Proof. See the Appendix for details not in the main text. ■

The decision to go ahead with the investment is postponed until timber prices have differentiated themselves clearly enough. This hysteresis leaves the land bare despite the fact that it would be unambiguously optimal to plant any of the two species immediately if the option to establish the other one was not available. In situations where the returns expected from a project, while high enough to warrant investment in isolation, might in the future be overtaken by the returns from an alternative project, using such a decision rule reduces the probability of finding out *ex post* that the least profitable option was chosen, irreversibly. Clearly the hysteresis may occur under any parameter combination satisfying Assumption 2, whether the drifts of the price processes are identical or not, and whether or not the chosen species is subject to uncertainty.

4. CHOOSING BETWEEN ALTERNATIVE REPLACEMENTS REPEATEDLY: AN EXTENSION OF FAUSTMAN'S PROBLEM

In order to keep the notation simple, some functions used in Section 3. will be redefined to account for multiple rotations; otherwise the notation is unchanged.

When only one species P is available, the expected value of harvesting after some time s a forest stand whose current age is a, and then replanting and harvesting the same species for an arbitrary number of rotations, is

$$G(p,a) = \max_s E e^{-rs}\left[p_s V(a+s) + F(p_s)\right] \tag{8}$$

where $F(p)$, the land value, solely depends on the current price of the single species available:

$$F(p) = \max_s E e^{-rs}\left[p_s V(s) + F(p_s)\right] \tag{9}$$

Thus, the land value coincides with the stand value at age zero as implied by the assumption that there are no planting or other investment costs; that is $F(p) = G(p,0)$.

When the timber price is constant, the optimal harvesting age a_F is constant from one rotation to the next and determined by Faustman's rule (1849); it is independent of the timber price and implicitly given by:

$$\frac{V_a(a_F)}{V(a_F)} = \frac{r}{1 - e^{-ra_F}} \tag{10}$$

The analysis has been extended to situations where the price may change over time (Brazee and Mendelsohn, 1988; Clarke and Reed, 1989; Morck et al. 1989; Thomson, 1992; Reed, 1993). When the price is governed by $(1a)$ and there are no fixed costs such as management costs the solution is equivalent to the deterministic solution with the timber value increasing at a constant exponential established by Newman et al. (1985).[6] This body of knowledge implies the following result:

Lemma 1 *When the timber price p follows the process* $(1a)$*, the land value* $F(p)$ *and the value of the stand at age a,* $G(p,a)$*, are homogenous of degree one in p, and the optimal cutting age is independent of the timber price p, precisely,*

$$\begin{aligned} G(p,a) &= ce^{\delta a}p && \text{(11a)} \\ F(p) &= cp && \text{(11b)} \\ c &= \frac{V(a_f)}{e^{\delta a_f} - 1} && \text{(11c)} \\ \frac{V_a(a_f)}{V(a_f)} &= \frac{\delta}{1 - e^{-\delta a_f}} && \text{(11d)} \end{aligned}$$

where a_f *is the optimal harvest age.*

Proof. See the appendix ■.

Both land and forest values $F(p)$ and $G(p,a)$ depend linearly on the current price of the unique tree species. The optimal harvest age is constant from one harvest to the next, thus independent of price, and equals the Faustman (constant price) rotation if and only if the drift of the stochastic timber price process is zero. We call it the generalized

---

[6] Saphores (2003) generalizes Faustman's formula to partial or total harvests in the case of a biomass whose stochastic growth is stock-dependent and represents the sole source of uncertainty. The optimal biomass at which harvest should occur is not a monotonic function of uncertainty, which implies that the relationship with Faustmann's rotation is ambiguous.

Faustman age in this paper. It is also smaller than the Wicksellian one-rotation optimal harvest age given by $(3c)$ as $\frac{V_a(a)}{V(a)}$ is decreasing in a.

We now turn to a situation not previously considered in the literature. Two tree species P and P' are available. The manager has the option to harvest and to replant the given tract of land with any combination of species, immediately after the harvest or after any delay, and to harvest again. The process goes on forever as establishing a new stand opens up the option to harvest, etc. As in the case of a single harvest, it can be shown that, to an expected value maximizer, any forest diversification strategy involving growing two species simultaneously would be dominated by one consisting in establishing one single species if at all. Thus suppose that the problem starts with trees of either species established on the forest tract. The manager may choose (i) to wait and observe the stand growing up, or (ii) to harvest the stand and replant the same species immediately, or (iii) to harvest the stand and plant the alternative species, or finally (iv) to harvest the stand and wait before planting one or the other species. As before we assume that there is no fixed cost associated with harvesting and establishing a new crop, so that the role of the alternative species is highlighted.

Consider the last possibility. After harvesting, the forest manager may wait before establishing a new crop. Then the land remains bare,[7] allowing the manager to wait until timber prices evolve in such a way that it is easier to choose the right species. No gain can be achieved by using that strategy, though. Indeed suppose the manager decides to establish one species at some date t strictly posterior to the harvest. Had she planted that species immediately after harvest, she would then be better off at t because the trees would have grown already. Had she planted the other species, she could cut the trees and plant the preferred species at no cost. In that case she would either be as well off, if cutting the existing trees produced no income, or she would be better off if cutting the existing trees produced some income. This proves the following proposition, which implies that only the first three possible decisions outlined in the past paragraph

---

[7] Allowing for costs of keeping the land bare, such as weeding or protection against erosion, would only reinforce the result.

need be considered.

Proposition 3 *It is optimal to establish a new crop immediately after harvesting.*

This result contrasts with the one indicating that it is optimal, in the Wicksellian two-species single-harvest case, to delay planting when the projects are not clearly differentiated. Indeed, the decision to choose a species that turns out to be undesirable *ex post* has a lower opportunity cost in case of multiple rotations: trees that turn out to be less desirable after planting may be cut and replaced with the desirable species. In the single rotation case, the possibility of a more profitable harvest is lost once the wrong species is established.

Consider now alternatives $(i) - (iii)$. Let $F(p, p')$ denote the value of the bare land, the value of the options to indefinitely plant and harvest the forest tract, choosing the prefered species P and P each time. Let $G(p, p', a)$ (respectively $G'(p', p, a)$) denote the value of the forest (land and trees) when the forest tract is populated with trees of species P (respectively P) of age a and price p (resp. p') while the price for the other species is p' (resp. p). Precisely,

$$G(p, p', a) = \max_s E e^{-rs} \left[p_s V(s+a) + F(p_s, p'_s) \right] \tag{12a}$$

$$G'(p', p, a) = \max_s E e^{-rs} \left[p'_s V'(s+a) + F(p_s, p'_s) \right] \tag{12b}$$

Lemma 2 *When p and p' follow the processes* (1*a*) *and* (1*b*) *respectively,* $F(p, p')$, $G(p, p', a)$, *and* $G'(p', p, a)$ *are homogenous of degree one in* (p, p') *and the optimal cutting age depends only on the current relative price of timber* $\theta = \frac{p'}{p}$. *Furthermore, the following reduced functions*

$$g(\theta, a) = \frac{1}{p} G(p, p', a) \tag{13a}$$

$$g'(\theta, a) = \frac{1}{p} G'(p', p, a) \tag{13b}$$

$$f(\theta) = \frac{1}{p} F(p, p') \tag{13c}$$

solve the following problems respectively

$$g(\theta, a) = \max_s \left[e^{-\delta s} V(a+s) + e^{-\delta s} E f(\theta_s)\right] \tag{14a}$$

$$g'(\theta, a) = \max_s \left[e^{-\delta' s} \theta V'(a+s) + e^{-\delta s} E f(\theta_s)\right] \tag{14b}$$

$$f(\theta) = \max_s \left[e^{-\delta s} V(s) + e^{-\delta s} E f(\theta_s)\right] \tag{14c}$$

Proof. See the appendix. ■

As a consequence of Lemma 3, it is certain that the trees being harvested are replaced immediately. At relative prices below some switching value θ^*, species P is planted if the land is bare; above θ^*, it is optimal to plant species P on a bare land. It is certain that θ^* exists as species P must be chosen when θ tends to zero and species P must be chosen when θ tends to infinity.

Proposition 4 *There exists a unique, strictly positive, value of θ, θ^* such that, if $\theta < \theta^*$ (respectively $\theta > \theta^*$) and the land is bare, it is optimal to plant species P (respectively species P) immediately, while indifferently planting P or P on bare land is optimal if $\theta = \theta^*$. Consequently, the reduced land value function is such that*

$$f(\theta) = g(\theta, 0) = g'(\theta, 0) \tag{15}$$

Furthermore, $f(\theta)$ is strictly increasing in θ with $\lim_{\theta \to 0} f(\theta) = c$, $\lim_{\theta \to +\infty} f(\theta) = c'\theta$, where $c = \frac{V(a_f)}{e^{\delta a_f} - 1}$ and $c' = \frac{V'(a'_f)}{e^{\delta' a'_f} - 1}$.

Proof. The set of $\theta > 0$ for which it is optimal to plant P is not empty as for θ positive and sufficiently small it is optimal to plant P. This set is bounded as for θ sufficiently high it is optimal to plant P. Being not empty and bounded, the set of $\theta > 0$ for which it is optimal to plant P has a finite maximum θ^*. The unicity of θ^* results from continuity.

The land value function $F(p, p')$ must be strictly increasing in p; then $F_p(p, p') > 0$. As $F_p(p, p') = f_\theta(\theta)$ then $f_\theta(\theta) > 0$ and therefore $f(\theta)$ is strictly increasing in θ. When θ tends to zero, that is when p' is close to zero while p is strictly positive with $p' \ll p$, P will remain the preferred species and the problem collapses to the single species case. Therefore, using $(11b)$, $\lim\limits_{\theta \to \infty} pf(\theta) = cp$ or $\lim\limits_{\theta \to 0} f(\theta) = c$. A similar proof shows that $\lim\limits_{\theta \to +\infty} f(\theta) = c'\theta$. In the continuation region, $G(p, p', a) = \max\limits_{s} E e^{-rs} \left[p_s V(a+s) + F(p_s, p'_s) \right]$, i.e. $G(p, p', 0) = F(p, p')$ or $f(\theta) = g(\theta, 0)$ if $(\theta, 0)$ belongs to the continuation region otherwise $G(p, p', 0) = F(p, p') + pV(0)$ then $f(\theta) = g(\theta, 0)$ as $V(0) = 0$. Similarly, one can show that $f(\theta) = g'(\theta, 0)$. ■

We are interested in characterizing the continuation region when species P is in place, and the continuation region when species P' is in place. In either case the continuation region is the set of points (θ, a) in $\mathbb{R}^+ \times \mathbb{R}^+$ where it is optimum to wait rather than exercise the option of harvesting.[8] Suppose that species P is in place, a situation which is desirable at relatively low values of θ. Clearly there is an age above which it is desirable to cut the trees so that the continuation region, if it exists, is bounded above (along the a dimension). Let $a^+(\theta)$ denote the upper boundary of the continuation region, the age above which it is optimum to harvest and below which it is desirable to allow the trees to grow further, given θ. At high enough values of θ, the alternative species P' becomes so attractive that there is no age at which it would be desirable to allow trees of species P to grow any further. At such high values of θ, the continuation region of species P does not exist. Let $\overline{\theta}$ denote the value of θ above which the continuation region does not exist for any harvest age and below which the continuation region exists for some age. Thus $\overline{\theta}$ signals the right-hand end of the continuation region.

It can be shown by contradiction that the continuation region exists for any $\theta < \overline{\theta}$. $\overline{\theta}$ is strictly higher than θ^*. Indeed, at $\theta = \theta^*$ the decision maker is indifferent between stands of either species P or P' populated with trees of age zero, which take a period $\underline{a}$ or $\underline{a}'$ before producing any valuable timber. Then if the stand is populated with trees of species P whose age is strictly between 0 and $\underline{a}$ while $\theta = \theta^*$, it is preferable to let them reach some age above $\underline{a}$, possibly allowing θ to become greater that θ^* before cutting them. This proves that $\theta^* \leq \overline{\theta}$. This simple argument also implies that, for values of the relative price in $\left[\theta^*, \overline{\theta}\right]$, it is optimal to allow the trees to continue growing if they

---

[8] While θ is the same variable in both continuation regions, a is specific to the species in place.

are older than some minimum age. Consequently, the continuation region has a lower boundary $a^{-}(\theta) > 0$ for $\theta \in \left[\theta^{*}, \bar{\theta}\right]$ as described in Figures 2 and 1.

A similar analysis applies to the continuation region when species P' is in place. Remembering that $\theta^{*\prime} \equiv \theta'$, it follows that $\bar{\theta}' \leq \theta^{*}$ and that the continuation region for P' has upper and lower boundaries $a^{+\prime}(\theta)$ on $[\bar{\theta}', +\infty[$, and $a^{-\prime}(\theta)$ on $\left[\bar{\theta}', \theta^{*}\right]$. Since the relative price θ is the same variable in both cases, only considered at different values, one should note that the expressions corresponding to each region are not the mirror image of each other, although the analysis obeys a symmetric logic. For example, obviously, $\bar{\theta}' \leq \theta^{*} \leq \bar{\theta}$ as species P' is desirable at high relative prices and the continuation region for P' does not exist below $\bar{\theta}'$ by definition of $\bar{\theta}'$.

Precisely, if species P is in place, then the upper boundary $a^{+}(\theta)$ is the smallest value of a such that

$$\arg\max_{s} e^{-\delta s}\left[V(s+a) + E\left(f(\theta_s)\right)\right] = 0. \tag{16}$$

implying that for $\theta \in \left[0, \bar{\theta}\right]$,

$$V_a\left(a^{+}(\theta)\right) - \delta V\left(a^{+}(\theta)\right) = \delta f(\theta) - \bar{\mu}\theta f_\theta(\theta) - \frac{1}{2}\bar{\sigma}^2\theta^2 f_{\theta\theta}(\theta) \tag{17}$$

The lower boundary $a^{-}(\theta)$ is defined by the conditions

$$\text{For } \theta \in \left[\theta^{*}, \bar{\theta}\right], \begin{cases} g(\theta, a) = V(a) + f(\theta), \text{ if } a = a^{-}(\theta) \\ g(\theta, a) > V(a) + f(\theta), \text{ if } a \in]a^{-}(\theta), a^{+}(\theta)[\\ g(\theta, a) < V(a) + f(\theta), \text{ if } a \in [0, a^{-}(\theta)[. \end{cases} \tag{18}$$

For $\theta \in \left[\theta^{*}, \bar{\theta}\right]$ and $a \in]a^{-}(\theta), a^{+}(\theta)[$, the condition $g(\theta, a) > V(a) + f(\theta)$ indicates that, if the land is populated with trees of species P and age a, it is preferable to allow them to reach maturity and harvest them at $a^{+}(\theta)$, rather than harvesting them immediately to obtain $V(a)$ plus the expected land value $f(\theta)$. For $\theta \in \left[\theta^{*}, \bar{\theta}\right]$ and $a \in [0, a^{-}(\theta)[$, the condition $g(\theta, a) < V(a) + f(\theta)$ indicates that it is optimal to harvest immediately.

For $\theta \in \left[0, \bar{\theta}\right]$, if it is optimal to wait and harvest later on, the forest value function $G(p, p', a)$ should satisfy Bellman's equation $E\left(dG(p, p', a)\right) = rG(p, p', a)da$ which can be written as the following partial differential equation governing the reduced forest value function $g(\theta, a)$ in the continuation region (proof in the appendix):

$$\frac{\bar{\sigma}^2}{2}\theta^2 g_{\theta\theta} + \bar{\mu}\theta g_\theta - \delta g + g_a = 0. \tag{19}$$

Thus, on the upper boundary $a^{+}(\theta)$ of the continuation region, the following value matching and smooth pasting conditions apply:

$$
\begin{aligned}
g\left(\theta, a^{+}(\theta)\right) &= V\left(a^{+}(\theta)\right) + f(\theta) && \text{(20a)}\\
g_{\theta}\left(\theta, a^{+}(\theta)\right) &= f_{\theta}(\theta) && \text{(20b)}\\
g_{a}\left(\theta, a^{+}(\theta)\right) &= V_{a}\left(a^{+}(\theta)\right) && \text{(20c)}
\end{aligned}
$$

The smooth pasting condition can be written in either of the two forms $(20b)$ or $(20c)$; given the value-matching condition these two forms are equivalent. It is usefull to combine the last four equations to derive Equation $(21a)$ linking the land value function $f(\theta)$ to the harvest age on the upper bounary $a^{+}(\theta)$, and Equation $(21b)$, applying when species P' is in place. Equation $(21c)$ is a combination of $(21a)$ and $(21b)$[9] which applies on the interval $\left[\overline{\theta}', \overline{\theta}\right]$ where both continuation regions exist as illustrated in Figure 3.

$$
\begin{aligned}
\frac{\overline{\sigma}^2}{2}\theta^2 f_{\theta\theta}(\theta) + \overline{\mu}\theta f_{\theta}(\theta) - \delta f(\theta) &= \delta V\left(a^{+}(\theta)\right) - V_{a}\left(a^{+}(\theta)\right), \theta \in \left[0, \overline{\theta}\right] && \text{(21a)}\\
\frac{\overline{\sigma}^2}{2}\theta^2 f_{\theta\theta}(\theta) + \overline{\mu}\theta f_{\theta}(\theta) - \delta f(\theta) &= \theta\left[\delta' V'\left(a^{+\prime}(\theta)\right) - V_{a}'\left(a^{+\prime}(\theta)\right)\right], \theta \in \left[\overline{\theta}', +\infty\right[&& \text{(21b)}\\
\delta V\left(a^{+}(\theta)\right) - V_{a}\left(a^{+}(\theta)\right) &= \theta\left[\delta' V'\left(a^{+\prime}(\theta)\right) - V_{a}'\left(a^{+\prime}(\theta)\right)\right], \theta \in \left[\overline{\theta}', \overline{\theta}\right] && \text{(21c)}
\end{aligned}
$$

Equation $(21a)$ is a non arbitrage condition and can be rearranged as $\frac{\overline{\sigma}^2}{2}\theta^2 f_{\theta\theta}(\theta) + \overline{\mu}\theta f_{\theta}(\theta) + V_{a}(a^{+}(\theta)) = \delta V(a^{+}(\theta)) + \delta f(\theta)$; then the right-hand side is the opportunity cost of waiting, per time unit; the first two terms on the left-hand side give the expected

---

[9] One can show that the reduced forest value function $g'(\theta, a)$ should satisfy the same partial differential equation as $g(\theta, a)$, that is

$$
\frac{\overline{\sigma}^2}{2}\theta^2 g'_{\theta\theta} + \overline{\mu}\theta g'_{\theta} - \delta g' + g_{a} = 0
$$

The boundary conditions are slightly different despite the symmetry. On the upper boundary $a^{+\prime}(\theta)$ of the corresponding continuation region, the following value matching and smooth pasting conditions apply:

$$
\begin{aligned}
g'\left(\theta, a^{+\prime}(\theta)\right) &= \theta V'\left(a^{+\prime}(\theta)\right) + f(\theta)\\
g'_{\theta}\left(\theta, a^{+\prime}(\theta)\right) &= V'\left(a^{+\prime}(\theta)\right) + f_{\theta}(\theta)\\
g'_{a}\left(\theta, a^{+}(\theta)\right) &= \theta V'_{a}\left(a^{+\prime}(\theta)\right)
\end{aligned}
$$

change in land value due to the variation of θ occuring during the same time unit where the presence of a second derivative arises from the stochasticity of θ; and the third term is the increase in the crop value due to timber growth per time unit.

The following proposition enunciates some additional properties of the upper and lower boundaries.

Proposition 5 *If species P is currently in place: For $\theta \in [0, \theta^*]$, the upper boundary $a^+(\theta)$ is such that $a^+(0) = a_f$ and there exists θ^1, $0 < \theta^1 < \overline{\theta}'$, such that $a^+(\theta)$ is decreasing on $\left[0, \theta^1\right]$ and increasing on $\left[\overline{\theta}', \theta^*\right]$. For $\theta \in \left[\theta^*, \overline{\theta}\right]$, $a^+(\theta)$ is decreasing while the lower boundary $a^-(\theta)$ is increasing with $a^-(\theta^*) = 0$. The upper and lower boundaries meet at $(\bar{\theta}, \underline{a})$. Similarly, if species P' is currently in place: For $\theta \in [\theta^*, +\infty[$, the upper boundary $a'^+(\theta)$ is such that $a^+(+\infty) = a'_f$ and there exists θ'^1, $\overline{\theta} < \theta'^1 < +\infty$, such that $a'^+(\theta)$ is increasing on $[\theta'^1 + \infty[$ and decreasing on $\left[\theta^*, \overline{\theta}\right]$. On $\left[\overline{\theta}', \theta^*\right]$, $a'^+(\theta)$ is increasing while the lower boundary $a'^-(\theta)$ is decreasing with $a'^-(\theta^*) = 0$. The upper and lower boundaries meet at $(\overline{\theta}', \underline{a}')$.*

Proof. When θ tends to zero, the problem collapses to the single species case analyzed above, for which the optimum harvest age is a_f given by (10) i.e. $a^+(0) = a_f$.

Equation (21a) implies that when θ is sufficiently close to zero, $\delta f(\theta) \simeq V_a(a^+(\theta)) - \delta V(a^+(\theta))$ where the land value function increases in θ while $V_a(a) - \delta V(a)$ decreases in a. It follows that $a^+(\theta)$ decreases in θ on some interval $\left[0, \theta^1\right]$ with $0 < \theta^1 < \theta^*$.

Let us show that the upper boundary $a^+(\theta)$ is strictly decreasing on $\left[\theta^*, \overline{\theta}\right]$. At (θ, a) on $a^+(\theta)$, the investor is indifferent between harvesting now and earning the left hand term of the following equation or waiting and harvesting later in which case she earns the right hand of the same equation:

$$V(a) + \max_s \left[e^{-\delta' s}\theta V'(s) + e^{-\delta s}Ef(\theta_s)\right] = \max_s \left[e^{-\delta s}V(a+s) + e^{-\delta s}Ef(\theta_s)\right]. \qquad (22)$$

Let a small period of time $da > 0$ elapse, over which θ becomes $\theta + d\theta$. The right-hand side (expected value of waiting) becomes

$\max_s \left[e^{-\delta s}V(a+s)+e^{-\delta s}Ef(\theta_s)+e^{-\delta s}daV_a(a+s)+e^{-\delta s}Edf(\theta_s)\right]$; the left-hand side (harvesting now) becomes

$$V\left(a\right)+daV_a\left(a\right)+\max_s\left[e^{-\delta' s}\theta V'(s)+e^{-\delta s}Ef(\theta_s)+e^{-\delta' s}d\theta V'(s)+e^{-\delta' s}\theta daV'_a(s)+\left(\delta-\delta'\right)dae^{-\delta' s}\theta V'(s)\right]$$

where the terms of order da are of a smaller order of magnitude than the terms in $d\theta$. Thus the effect of a small move away from the upper boundary $a^+\left(\theta\right)$ can be approximated while neglecting terms in da: equality (22) will be broken or maintained depending on whether$V\left(a\right)+\max_s\left[e^{-\delta' s}\theta V'(s)+e^{-\delta s}Ef(\theta_s)+e^{-\delta' s}d\theta V'(s)\right]$ is higher than, lower than, or equal to $\max_s\left[e^{-\delta s}V(a+s)+e^{-\delta s}Ef(\theta_s)+e^{-\delta s}Edf(\theta_s)\right]$. Using (22) and noting that $Edf(\theta_s)=f_\theta Ed\theta_s+\frac{1}{2}f_{\theta\theta}Ed\theta_s^2$ is of an order of magnitude smaller of equal to da, this reduces to evaluating the sign of $e^{-\delta' s}d\theta V'(s)$: positive if $d\theta>0$ and *vice versa*. This means that a small motion to the right from the boundary $a^+\left(\theta\right)$ causes the pair (θ,a) to fall into the immediate harvest zone. In turn, this implies that the boundary is downward sloping.

Let us show by the same method that the lower boundary $a^-\left(\theta\right)$ is increasing on $\left[\theta^*,\overline{\theta}\right]$. At (θ,a) on $a^-\left(\theta\right)$ with $a<\underline{a}$, the investor is indifferent between harvesting immediately thus earning the land value (the harvest is worth zero) given by the left-hand term of the following equation, or harvesting later at age $a+s>\underline{a}$, in which case she earns the right hand of the same equation:

$$\max_s\left[e^{-\delta' s}\theta V'(s)+e^{-\delta s}Ef(\theta_s)\right]=\max_s\left[e^{-\delta s}V(a+s)+e^{-\delta s}Ef(\theta_s)\right]. \qquad (23)$$

After a small time interval da, ignoring terms of order of magnitude smaller than or equal to da, the difference between the left-hand and the right-hand side of (23) becomes $e^{-\delta' s}d\theta V'(s)$: positive if $d\theta>0$ and *vice versa* as above. This means that a small motion to the right from the boundary $a^-\left(\theta\right)$ causes the pair (θ,a) to fall into the immediate harvest zone. Since the immediate harvest zone is below the frontier, this implies that the lower boundary is upward sloping.

Note that at $(\theta^*,0)$ the investor is indifferent between planting species P or switching to species $P\prime$; thus $(\theta^*,0)$ belongs to $a^-\left(\theta\right)$.

By symmetry, if species $P\prime$ is in place then $a'^+\left(\theta\right)$ is strictly increasing while the lower boundary $a'^-(\theta)$ is strictly decreasing on $\left[\overline{\theta}', \theta^*\right]$ as depicted on Figure 3. Equation (21c) linking $a^+(\theta)$ and $a'^+(\theta)$ on $\left[\overline{\theta}', \overline{\theta}\right]$ shows that as $a^+(\theta)$ is increasing then $a'^+(\theta)$ must also be increasing on the same interval.

Let us show that $a^+\left(\overline{\theta}\right) = a^-\left(\overline{\theta}\right) = \underline{a}$. Suppose as a proposition to be contradicted, that $a^+\left(\overline{\theta}\right) = \underline{a} + \varepsilon$, $\varepsilon > 0$. Then $V(\underline{a} + \varepsilon) + f(\bar{\theta}) > V(\underline{a}) + f(\bar{\theta}) \geq V(a^-\left(\overline{\theta}\right)) + f(\bar{\theta})$ where the left-hand side applies the definition (16) of the upper boundary while the right-hand side results from the condition that $a^-\left(\overline{\theta}\right) \leq \underline{a}$. But then, for some pair $\left(\varepsilon', \overline{\theta}'\right)$, $0 < \varepsilon' < \varepsilon$; $\overline{\theta}' > \bar{\theta}$, it is also true that $V(\underline{a} + \varepsilon') + f(\bar{\theta}') > V(a^-\left(\overline{\theta}\right)) + f(\bar{\theta}')$ so that $\bar{\theta}$ is not the maximum value of θ at which the continuation region exists, contradicting its definition. This contradiction can be avoided only if $\varepsilon = 0$, implying that $a^+\left(\overline{\theta}\right) = \underline{a}$. Now suppose as a proposition to be contradicted that $a^-\left(\overline{\theta}\right) < \underline{a}$; then by (18), $g\left(\overline{\theta}, a^-\left(\overline{\theta}\right)\right) > V\left(a^-\left(\overline{\theta}\right)\right) + f(\bar{\theta}) = f(\bar{\theta})$. Since $g\left(\overline{\theta}, \underline{a}\right) > g\left(\overline{\theta}, a^-\left(\overline{\theta}\right)\right)$ if $a^-\left(\overline{\theta}\right) < \underline{a}$, there exists $\overline{\theta}' > \overline{\theta}$ such that $g\left(\overline{\theta}', \underline{a}\right) > g\left(\overline{\theta}', a^-\left(\overline{\theta}\right)\right)$ so that $\bar{\theta}$ is not the maximum value of θ at which the continuation region exists. This contradiction can be avoided only if $a^-\left(\overline{\theta}\right) \geq \underline{a}$; since $a^-\left(\overline{\theta}\right) \leq \underline{a}$, it follows that $a^-\left(\overline{\theta}\right) = \underline{a}$. We conclude then that the upper and lower boundaries meet at $(\bar{\theta}, \underline{a})$.

■

To understand these results, recall that in Faustman's model with a single species and positive planting cost, rotations are shorter when planting cost decreases or the forest's value increases. Here, when θ increases from 0, the harvest age decreases and becomes lower than Faustman optimum age (shorter rotation) because the alternative species is an option (forest with higher value) while choosing is easy, meaning higher forest value than single species, at low cost in terms of commiting an error in choosing the valuable species. On the contrary, when θ is closer to θ^*, the switching cost becomes higher (it is not easy to make a choice) and consequently rotations are increasingly longer.

4.1 Analytical resolution under certainty

We now assume that $\sigma = \sigma' = 0$. Prices are certain, although not necessarily constant. While this is a particular case of the stochastic version of the problem, it can be solved analytically entirely.

If $\mu = \mu'$ then the relative price θ remains constant. If a species is in place optimally then it will continue to be planted and harvested successively forever. Switching from one species to the alternative one can not occur unless the problem starts with an inherited stand that should not have been planted in the first place.

When $\mu \neq \mu'$ switches are possible. Without loss of generality, assume that $\overline{\mu} = \mu' - \mu > 0$; the drift of the price of the currently planted species P is smaller than the drift of the price of the alternative species P. Consider a stand of species P at age a when its price is p and the price of the alternative species p' is still relatively low. The decision maker will cut this stand and replant the same species and so on until the price of the alternative species P has sufficiently increased with respect to the price of the species currently planted. Suppose that species P is replanted n times until the switch to P occurs. At that moment, species P will be cut for the last time and species P will be planted thereafter forever. Therefore, by analogy with $(11b)$, the land value at that moment will be $c'p'$. It will be convenient to index cutting ages and the corresponding prices in reverse chronological order. Thus a_0 is the age at which species P is cut for the last time. Cutting ages a_n, a_{n-1},..,a_0 will be chosen to maximize the stand value $G(p, p', a) = pg(\theta, a)$, that is

$$\begin{aligned} G(p,p',a) &= \max_{a_0,\ldots a_n} \left[p_n V(a_n) e^{-r(a_n - a)} + \ldots \right. \\ & \left. + p_0 V(a_0) e^{-r(a_n+\ldots+a_0-a)} + c' p'_0 e^{-r(a_n+\ldots+a_0-a)}\right] \end{aligned}$$

where $p_i = pe^{\mu(a_n+\ldots+a_i-a)}$ for $i = 0, \ldots n$ and $p'_0 = p' e^{\mu'(a_n+\ldots+a_0-a)}$. This implies that

$$\begin{aligned} G(p,p',a) &= p \max_{a_0,\ldots a_n} \left[V(a_n) e^{-\delta(a_n-a)} + \ldots + V(a_0) e^{-\delta(a_n+\ldots+a_0-a)} \right. \\ & \left. + c' \frac{p'}{p} e^{-\delta'(a_n+\ldots+a_0-a)}\right] \end{aligned}$$

so that

$$\begin{aligned} g(\theta,a) &= \max_{a_0,\ldots a_n} \left[\left(V(a_n)e^{-\delta a_n} + \ldots + V(a_0) e^{-\delta(a_n+\ldots+a_0)}\right) e^{\delta a} \right. \\ & \left. + c'\theta e^{-\delta'(a_n+\ldots+a_0)} e^{\delta' a}\right] \end{aligned} \quad (24)$$

For the purpose of next sections, consider the two decreasing and convex functions $K(a) = \frac{V_a(a)}{\delta} - V(a)$ and $L(a) = \frac{V_a(a)}{\delta e^{\delta a}}$ crossing each other at Faustman's age a_f as illustrated in Figure 5.

Proposition 6 *It exists $\theta_0 > 0$ such that the manager is indifferent between planting on a bare land species P' forever or species P for one rotation of length $\underline{a}_0$ and then species P' forever. Both θ_0 and $\underline{a}_0$ are determined by*

$$\begin{aligned} \frac{V_a(\underline{a}_0)}{V(\underline{a}_0)} &= \delta + \frac{\delta'}{e^{\delta' \underline{a}_0} - 1} & \text{(25a)} \\ K(\underline{a}_0) &= \frac{\delta'}{\delta} c' \theta_0 e^{\overline{\mu} \underline{a}_0} & \text{(25b)} \end{aligned}$$

Besides, $\underline{a}_0$ satisfies $\underline{a} < \underline{a}_0 < a_f$.

Proof. The existence of θ_0 and $\underline{a}_0$ stems from Proposition 5. As by definition the manager is indifferent between planting P' forever or planting P for just one rotation of length $\underline{a}_0$ then $c'\theta_0 = \max_s \left[V(s)e^{-\delta s} + c'\theta_0 e^{-\delta' s}\right]$ where $\underline{a}_0 = \arg\max_s \left[V(s)e^{-\delta s} + c'\theta_0 e^{-\delta' s}\right]$. The first order condition of this maximization problem is equivalent to equation $(25b)$. The second order condition is $K_a(\underline{a}_0) - \overline{\mu} K(\underline{a}_0) < 0$, satisfied for $\underline{a}_0 \in]\underline{a}, a_w[$ as $V(a)$ is decreasing and concave on this interval and $\overline{\mu} > 0$. The first order condition, together with $c'\theta_0 = V(\underline{a}_0)e^{-\delta a_0} + c'\theta_0 e^{-\delta' \underline{a}_0}$, gives Equation $(25a)$ that allows to determine $\underline{a}_0$ unambiguously whereas $(25b)$ determines θ_0. As $\frac{V_a(a)}{V(a)}$ is decreasing on $[\underline{a}, +\infty[$ and $\lim_{a \to \underline{a}} \frac{V_a(a)}{V(a)} = +\infty$ then $\underline{a}_0 > \underline{a}$. Faustman's age a_f is determined by Equation $(11d)$ that is equivalent to $\frac{V_a(a_f)}{V(a_f)} = \delta + \frac{\delta}{e^{\delta a_f} - 1}$. One can prove that $\frac{\delta}{e^{\delta a} - 1} < \frac{\delta'}{e^{\delta' a} - 1}$ for any $a > 0$ as $\delta' < \delta$. Consequently, $\frac{V_a(a_f)}{V(a_f)} < \frac{V_a(\underline{a}_0)}{V(\underline{a}_0)}$, thus $\underline{a} < \underline{a}_0 < a_f$. ■

When $\theta > \theta_0$, it is conceivable that a stand of age a of species P may be inherited by the decision maker while not resulting from a rational decision to establish it. Depending on the stand age and on the value of θ, the decision maker may be better off harvesting immediately and planting the alternative species, or letting the stand reach maturity before switching.

Proposition 7 *For $\theta \geq \theta_0$, the upper switching boundary $a^+(\theta)$ is defined analytically as*

$$\left\{(\theta, a)/\ \theta_0 \leq \theta \leq \bar{\theta},\ \underline{a} \leq a \leq \overline{a},\ \text{and } K(a) = c' \frac{\delta'}{\delta} \theta \right\} \tag{26}$$

where $\bar{\theta}$ and $\overline{a}$ are respectively the unique solutions to $V_a(\underline{a}) = \delta' c' \bar{\theta} e^{\overline{\mu} \underline{a}}$ and $K(\overline{a}) = c' \frac{\delta'}{\delta} \theta_0$. It is a decreasing curve in the plan (θ, a) stretching between $(\theta_0, \overline{a})$ and $(\bar{\theta}, \underline{a})$.

Proof. Consider an inherited stand P characterized by (θ, a) such that with $\theta \geq \theta_0$ and $a \geq \underline{a}$. The maximization problem is

$$\max_s \left[V(s) e^{-\delta s} + c' \theta e^{-\delta' s} \right]$$

whose first order condition is equivalent to $K(a) = c' \frac{\delta'}{\delta} \theta$ whereas the second order condition is $K_a(a) - \overline{\mu} K(a) < 0$, satisfied for $a \in]\underline{a}, \overline{a}[$ as $V(a)$ is decreasing and concave on this interval and $\overline{\mu} > 0$. The first order condition defines a decreasing curve $a(\theta)$. The proof of monotonicity is immediate since $K(a) = \frac{V_a(a)}{\delta} - V(a)$ is decreasing in a. The highest value of θ compatible with $a \geq \underline{a}$ defines $\bar{\theta}$ with $K(\underline{a}) = \frac{V_a(\underline{a})}{\delta} = \frac{\delta'}{\delta} c' \bar{\theta} e^{\overline{\mu} \underline{a}}$ or $V_a(\underline{a}) = \delta' c' \bar{\theta} e^{\overline{\mu} \underline{a}}$. The highest value of a is $\overline{a}$ compatible with θ_0 with $\overline{a} < a_w$ as $K(\overline{a}) = c' \frac{\delta'}{\delta} \theta_0 > K(a_w) = 0$. ■

Proposition 8 *For $\theta \in [\theta_0, \bar{\theta}]$, the lower switching boundary of the waiting region $a^-(\theta)$ is defined analytically as*

$$\begin{aligned} &\{(\theta, a)/\ \theta_0 \leq \theta \leq \bar{\theta},\ 0 \leq a \leq \underline{a}, \exists s \geq 0/ \\ &\quad \left(1 - e^{-\delta' s}\right) c' \theta = e^{-\delta s} V(a+s) \text{ and } K(a+s) = \frac{\delta'}{\delta} c' \theta e^{\overline{\mu} s} \Big\} \end{aligned} \tag{27}$$

It is an increasing curve in the plan (θ, a) stretching between $(\theta_0, 0)$ and $(\bar{\theta}, \underline{a})$.

Proof. For $a \in [0, \underline{a}]$, the set of points (θ, a) for which the decision maker is indifferent between harvesting immediately to earn $p\left[c'\theta + V(a)\right]$ or harvesting after a time period s maximizing $e^{-rs}[p_s V(a+s) + c' p'_s]$ defines the lower switching boundary of the waiting region. It is defined as the set of points (θ, a) solution to $c'\theta + V(a) = e^{-\delta s} V(a+s) + e^{-\delta' s} c' \theta$ and $K(a+s) = \frac{\delta'}{\delta} c' \theta e^{\overline{\mu} s}$ where $V(a) = 0$, $\underline{a} - a \leq s \leq \underline{a}_0 - a$, and $a + s \in [\underline{a}, \underline{a}_0]$

is the age at which the trees will be cut and a is their age on the lower boundary. Pairs (θ, a) below or to the right of the lower boundary command immediate cutting; while pairs above the lower boundary but below the upper boundary belong to the continuation region. The two equations in Proposition 8 lead to $\frac{V_a(a+s)}{V(a+s)} = \delta + \frac{\delta'}{e^{\delta' s}-1}$. For a given $a \in [0, \underline{a}]$, $\frac{V_a(a+s)}{V(a+s)}$ decreases in $s \in [\underline{a} - a, \underline{a}_0 - a]$ from $+\infty$ to $\frac{V_a(\underline{a}_0)}{V(\underline{a}_0)}$ whereas $\delta + \frac{\delta'}{e^{\delta' s}-1}$ decreases from a finite positive value $\delta + \frac{\delta'}{e^{\delta'(\underline{a}-a)}-1}$ to $\delta + \frac{\delta'}{e^{\delta'(\underline{a}_0-a)}-1} > \delta + \frac{\delta'}{e^{\delta' \underline{a}_0}-1} = \frac{V_a(\underline{a}_0)}{V(\underline{a}_0)}$ by (25a). Therefore, for any $a \in [0, \underline{a}]$, there exists a unique $s \in [\underline{a} - a, \underline{a}_0 - a]$ such that $\frac{V_a(a+s)}{V(a+s)} = \delta + \frac{\delta'}{e^{\delta' s}-1}$ that we denote $s(a)$. Furthermore, as $\frac{V_a(a+s)}{V(a+s)}$ decreases in a, then $s(a)$ decreases in a, too. The second equation $K(a+s) = \frac{\delta'}{\delta} c' \theta e^{\overline{\mu} s}$ shows that for any $a \in [0, \underline{a}]$, there exists a unique $\theta = \theta(a)$ satisfying the condition in Proposition 8. To show that $\theta_a(a) < 0$, derive $\left(1 - e^{-\delta' s}\right) c'\theta = e^{-\delta s} V(a+s)$ with respect to a to obtain $\left(1 - e^{-\delta' s}\right) c'\theta_a = -\delta e^{-\delta s} V(a+s) s_a + e^{-\delta s}(1 + s_a) V_a(a+s) - \delta' e^{-\delta' s} c' \theta s_a$ or

$$\left(1 - e^{-\delta' s}\right) c'\theta_a = -\delta e^{-\delta s} V(a+s) s_a + e^{-\delta s}(1+s_a) V_a(a+s) - \delta e^{-\delta s} s_a \left(\frac{V_a(a+s)}{\delta} - V(a+s)\right)$$

that is

$$\left(1 - e^{-\delta' s}\right) c'\theta_a = e^{-\delta s} s_a V_a(a+s). \text{ As } s_a(a) < 0 \text{ then } \theta_a(a) < 0.$$

The point $(\theta_0, 0)$ satisfies the condition in Proposition 8 with $s = \underline{a}_0$ and $a = 0$. The point $(\bar{\theta}, \underline{a})$ satisfies this condition with $s = 0$ and $a = \underline{a}$. The lower switching boundary $a^-(\theta)$ is then an increasing curve in the plan (θ, a) stretching between $(\theta_0, 0)$ and $(\bar{\theta}, \underline{a})$.

■

An implication of the two last propositions is that there does not exist any continuation zone for values of θ above $\overline{\theta}$. If a stand P exists for such values, it must be harvested immediately and the alternative tree species must be established.

Proposition 9 *Assume that $\mu < \mu'$ and the land is occupied by the stand P characterized by the pair (θ, a) with $\theta < \theta_0$. If the manager has not to harvest the stand immediately then she must optimally harvest it and replant the land with the same species P at pairs $(\theta e^{\mu(a_n + \ldots + a_k - a)}, a_k)$, $k = 1, \ldots, n$, and finally harvest for the last time the stand P at pair $(\theta e^{\overline{\mu}(a_n + \ldots + a_0 - a)}, a_0)$ and immediately switch once and forever to species P'. The* *cutting ages* $(a_n, ..., a_{1,} a_0)$ *can be determined by the following* $n+1$ *equations.*

$$K(a_k) = L(a_{k-1}), \textit{ for } k = 1, ..., n \tag{28a}$$

$$K(a_0) = \frac{\delta'}{\delta} c' \theta e^{\overline{\mu}(a_n + ... + a_0 - a)} \tag{28b}$$

where the number of rotations $n+1$ *is determined endogenously to satisfy* $\theta e^{\overline{\mu}(a_n + ... + a_0 - a)} \geq \theta_0$ *and* $\theta e^{\overline{\mu}(a_n + ... + a_1 - a)} \leq \theta_0$.

Proof. Note that the solution $(a_n, ..., a_0)$ to the maximization problem corresponding to $g(\theta, a)$ is the same as the one corresponding to $g(\theta_0, 0)$ where $\theta_0 e^{(\mu' - \mu)a} = \theta$. As so, it is possible to set a to zero in the maximization problem. Consider then the maximization problem $g(\theta, 0) = \max_{a_0, ... a_n} W(\theta, a_n, ..., a_0)$ where $W(\theta, a_n, ..., a_0)$ denotes $V(a_n) e^{-\delta a_n} + ... + V(a_0) e^{-\delta(a_n + ... + a_0)} + c' \theta e^{-\delta'(a_n + ... + a_0)}$. The $n+1$ first order conditions of this maximization problem are $V_a(a_k) e^{-\delta(a_k + ... + a_n)} - \delta \sum_{i=0}^{k} V(a_i) e^{-\delta(a_i + ... + a_n)} - \delta' c' \theta e^{-\overline{\mu} a} e^{-\delta'(a_0 + ... + a_n)} = 0$ for $k = 0, ..., n$. This set of $n+1$ equations can be equivalently simplified into another set of $n+1$ equations obtained by keeping the 1^{st} order condition for $k = 0$, and for $k = 1, ..., n$, substituting the k^{th} first order condition for the equation obtained by subtracting the $(k-1)^{th}$ first order condition from the k^{th} one. This transformation of the $n+1$ first order conditions of the maximization problem leads to the following equivalent set of $n+1$ equations $V_a(a_0) - \delta V(a_0) - \delta' c' \theta e^{\overline{\mu}(a_0 + ... + a_n - a)} = 0$ or $K(a_0) = \frac{\delta'}{\delta} c' \theta_0$, and $V_a(a_k) e^{-\delta(a_k + ... + a_n)} - \delta V(a_k) e^{-\delta(a_k + ... + a_n)} = V_a(a_{k-1}) e^{-\delta(a_{k-1} + ... + a_n)}$ or $K(a_k) = L(a_{k-1})$ for $k = 1, ..., n$. We show in the appendix that the second order condition is satisfied. ■

Conditions $(28a)$ apply at harvests where it is optimal to replant the same species; they can be called replanting conditions. The last condition $(28b)$ applies when switching to the alternative species is optimal, upon the last harvest of species P. A consequence of Proposition 9 is the existence of a strictly decreasing sequence $(\theta_k)_{k \in \mathbb{N}}$ with first term θ_0 and $\lim_{n \to +\infty} \theta_n = 0$ such that if the land is bare and relative timber price is $\theta \in [\theta_{n+1}, \theta_n]$ than it is optimal to plant it with species P exactly n times before switching definitely to species P'. As illustrated in Figure 6, when θ varies in the interval $[\theta_{n+1}, \theta_n]$, let $[\underline{a}_n, \overline{a}_n]$ denote the interval of the corresponding cutting age.

For the purpose of next propositions, let $R(a)$ denotes the unique positive number in $]\underline{a}, +\infty[$ associated to $a \in]\underline{a}, +\infty[$ such that $K(R(a)) = L(a)$. As illustrated in Figure 5, on the interval $]\underline{a}, +\infty[$, the function R has the following proprieties

$$\begin{aligned}
\forall a &\in]\underline{a}, +\infty[, \; R(a) = a \Longleftrightarrow a = a_f \\
\forall a &\in]\underline{a}, +\infty[, \; R(a) < a \Longleftrightarrow a > a_f \\
\forall a &\in]\underline{a}, +\infty[, \; R(a) > a \Longleftrightarrow a < a_f
\end{aligned}$$

Proposition 10 *The boundary $a(\theta)$ of the continuation region for $\theta \leq \theta_0$ is the union of infinity of decreasing and discontinuous curves $a_n(\theta)$ stretching between $(\theta_{n+1}e^{\overline{\mu}\ \overline{a}_n}, \overline{a}_n)$ and $(\theta_n e^{\overline{\mu}\ \underline{a}_n}, \underline{a}_n)$ with $\underline{a}_{n+1} < \overline{a}_n$, $a_f \in [\underline{a}_{n+1}, \overline{a}_{n+1}] \subset [\underline{a}_n, \overline{a}_n]$, for $n = 0, ..., +\infty$. Precisely, sequences $(\underline{a}_n)_{n\in\mathbb{N}}$ and $(\overline{a}_n)_{n\in\mathbb{N}}$ are respectively increasing and decreasing with $\lim_{n\to+\infty} \underline{a}_n = \lim_{n\to+\infty} \overline{a}_n = a_f$.*

Proof. Assume that $\mu < \mu'$. When the relative timber price θ varies in $[\theta_1, \theta_0]$, by definition of θ_0 and θ_1, it is optimal to plant a bare land with species P and cut the stand at age $a(\theta)$ such that $K(a(\theta)) = \frac{\delta'}{\delta}c'\theta e^{\overline{\mu}a(\theta)}$ where $\theta e^{\overline{\mu}a(\theta)} > \theta_0$ and switch immediately to species P'. Recall that, by Proposition 7, $a^+(\theta)$ is a decreasing curve in the plan (θ, a) stretching between $(\theta_0, \overline{a})$ and $(\overline{\theta}, \underline{a})$. By definition of θ_1, when the relative timber price is equal to θ_1, the manager is indifferent between planting species P and harvest later on at age $\overline{a}_0$ and switching to species P', or harvesting and replanting the same species for the last time at age $\underline{a}_1$. Note that $\overline{a}_0 \geq \underline{a}_1$, by definition of $\overline{a}_0$, $\underline{a}_1$, and θ_0. Assume that $\overline{a}_0 = \underline{a}_1$, that is the harvest will take place at some time in the future at which the manager is indifferent between species P or P' to be planted. Then, necessarily $\overline{a}_0 = \underline{a}_1 = \overline{a}$, $\theta_0 = \theta_1$, and $K(\overline{a}) = L(\underline{a}_0)$ and $K(\overline{a}) = \frac{\delta'}{\delta}c'\theta_0$ by Proposition 9. As $K(\underline{a}_0) = \frac{\delta'}{\delta}c'\theta_0 e^{\overline{\mu}\underline{a}_0}$ then, $L(\underline{a}_0) = K(\underline{a}_0)e^{\overline{\mu}\underline{a}_0}$ implying $\frac{V_a(\underline{a}_0)}{V(\underline{a}_0)} = \delta + \frac{\delta}{e^{\delta'\underline{a}_0} - 1}$ that contradicts equation $(25a)$ as long as $\delta \neq \delta'$. We conclude that $\overline{a}_0 > \underline{a}_1$.

Consider now the cutting boundary $a_1(\theta)$ corresponding to a stand that has been planted with species P at date 0 when timber relative price was $\theta \in [\theta_1, \theta_0]$. Consider two pairs $(xe^{\overline{\mu}a(x)}, a(x))$ and $(ye^{\overline{\mu}a(y)}, a(y))$ of this curve such that $x < y$. There exist only two pairs $(x'e^{\overline{\mu}a(x')}, a(x'))$ and $(y'e^{\overline{\mu}a(y')}, a(y'))$ on the upper switching curve $a^+(\theta)$ such that $R(a(x)) = a(x')$, $R(a(y)) = a(y')$, $x' < y'$, and $a(x') > a(y')$ as it is decreasing in the plan (θ, a), as depicted in Figure 5. One can show that the function R conserves the order relationship and therefore necessarily $a(x') < a(y')$. We conclude that $a_1(\theta)$ is a decreasing curve in the plan (θ, a). The proof can be recursively repeated to show that all curves $a_n(\theta)$, $n = 2, ... + \infty$, are decreasing in the plan (θ, a).

For $n = 0, ..., +\infty$, by definition of $a_{n+1}(\theta)$, for any pair $(x, a(x)) \in a_{n+1}(\theta)$, there exists a unique pair $(x', a(x')) \in a_n(\theta)$ such that $R(a(x)) = a(x')$. Given the proprieties of the function R, this implies that $[\underline{a}_{n+1}, \overline{a}_{n+1}] \subset [\underline{a}_n, \overline{a}_n]$

and $\lim_{n \to +\infty} \underline{a}_n = \lim_{n \to +\infty} \overline{a}_n = a_f$.

■

Proposition 11 *When $\mu < \mu'$, the reduced land value function $f(\theta)$ and the corresponding cutting age can be computed recursively using*

$$f(\theta) = \max_{\theta' \geq \theta} \left(\frac{\theta'}{\theta}\right)^{-\frac{\delta}{\delta - \delta'}} \left[V\left(\frac{\log(\frac{\theta'}{\theta})}{\mu' - \mu}\right) + f(\theta')\right] \tag{29}$$

for $\theta \leq \theta_0$ and using $f(\theta) = b'\theta$ for $\theta \geq \theta_0$.

Proof. When the current prices are given by p and $\theta \leq \theta_0$, it is optimal to plant species P and the land value function is defined as $pf(\theta) = \max_{s \geqslant 0} e^{-rs} [p_s V(s) + p_s f(\theta_s)]$. Under certainty, $p_s = pe^{\mu s}$ and $\theta_s = \theta e^{(\mu' - \mu)s} = \theta e^{(\delta - \delta')s}$. Consequently, $f(\theta) = \max_{s \geqslant 0} e^{-\delta s} [V(s) + f(\theta_s)]$ where s is replaced by $\frac{1}{\mu' - \mu} \log(\frac{\theta_s}{\theta})$, to obtain the expression of $f(\theta)$ in Proposition 11. ■

Note that the cutting age as a function of θ is a discontinuous curve. The discontinuous part corresponds to $\theta \leq \theta_0$, that is the replanting part of the cutting age. The discontinuity is present to distinguish between a stand that has to be replanted for n times with species P before switching definitely to species P and another who has to be replanted exactly $n - 1$ times before switching to species P.

Proposition 11 is used to compute the land value function under certainty. At relative prices equal to or higher than the switching threshold θ_0, land value equals $c'\theta$, the land value function that applies when the sole species is P.

4.2 Numerical resolution with uncertainty

In order to conclude the description of the solution of the model with uncertainty, in particular in order to describe the boundary of the continuation region, it is necessary to use numerical methods. The reduced forest value function $g(\theta, a)$ must satisfy the partial differential equation (19) and the value-matching and smooth pasting conditions (Equations (20a), (20b), and (20c)). The partial differential equation governing $g(\theta, a)$ can be simplified by performing the change of variable $x = \log(\theta)$. Let $h(x, a) = g(\theta, a)$ and $l(x) = f(\theta)$, then the partial differential equation governing $h(x, a)$ is

$$\frac{\overline{\sigma}^2}{2} h_{xx} + \left(\overline{\mu} - \frac{\overline{\sigma}^2}{2}\right) h_x - \delta h + h_a = 0 \tag{30}$$

The optimal stopping problem of valuing the forest value when species P is planted is similar to the problem of valuing an American-type option with free boundary. Because the free boundary location is not known in advance, the value-matching and smooth pasting conditions cannot be of immediate help. These conditions can be used to localize the free cutting boundary once the stand value is determined. We do so iteratively as explained later in this section. To compute the forest value function $h(x, a)$, it is helpful to specify the corresponding optimal stopping problem as a linear complementarity one (Zvan et al, 1998). Let $\mathcal{L}$ be the linear operator defined as

$$\mathcal{L}.h = \frac{\overline{\sigma}^2}{2} h_{xx} + \left(\overline{\mu} - \frac{\overline{\sigma}^2}{2}\right) h_x - \delta h + h_a$$

Then, the linear complementarity problem is

$$\begin{aligned} \mathcal{L}.h(x, a) &\geq 0 \\ h(x, a) - (l(x) + V(a)) &\geq 0 \\ \mathcal{L}.h(x, a) \left[h(x, a) - (l(x) + V(a))\right] &= 0 \end{aligned}$$

Note that this formulation does not imply any explicit use of the free cutting boundary $a(\theta)$. It shows that the value function $h(x, a)$ can be considered as the value of an

American option with expiry date equals to infinity, an underlying asset which is a geometric diffusion process with drift $\overline{\mu}$ and volatility $\overline{\sigma}^2$, and a discount factor equal to δ. Indeed, in the continuation region where it is optimal to continue holding the option to harvest, the required return δh is equal to the actual return or equivalently $\mathcal{L}.h(x,a) = 0$ and the option value is higher than the payoff, that is $h(x,a) - (l(x) + V(a)) > 0$. Consequently, it is not yet optimal to exercise. In the stopping region, it is no more optimal to continue holding the option to harvest but it is optimal to harvest immediately because the required return δh is less than the actual return or equivalently $\mathcal{L}.h(x,a) > 0$ and the option value must equal the payoff that is $h(x,a) - (l(x) + V(a)) = 0$. The free cutting boundary is just where the decision maker is indifferent between harvesting immediately or continuing to hold this option, that is when $\mathcal{L}.h(x,a) = 0$ and $h(x,a) - (l(x) + V(a)) = 0$.

When valuing an American option, the use of the complementarity formulation is straightforward. The option value is then computed numerically by performing a discretization of the linear complementarity problem. The main difference between valuing an American option and the forest value is that the payoff in the former case is known as a function of the underlying asset when exercising the option whereas it should be endogenously determined in the latter as it is the sum of the timber crop value and the land value $l(x) = h(x,0)$. For this reason, valuing the forest value function $h(x,a)$ as a complementarity problem as specified above is seemingly not possible. Instead, we need to consider simultaneously the second problem consisting in valuing the forest value function $g'(\theta,a) = h'(x,a)$ as follows:

$$\mathcal{L}.h(x,a) \geq 0 \tag{31a}$$

$$h(x,a) - (l(x) + V(a)) \geq 0 \tag{31b}$$

$$\mathcal{L}.h(x,a)\,[h(x,a) - (l(x) + V(a))] = 0 \tag{31c}$$

$$\begin{aligned}\mathcal{L}.h'(x,a) &\geq 0 \qquad (32a)\\ h'(x,a)-(l(x)+e^xV(a)) &\geq 0 \qquad (32b)\\ \mathcal{L}.h'(x,a)\,[h'(x,a)-(l(x)+e^xV(a))] &= 0 \qquad (32c)\end{aligned}$$

$$l(x)=h(x,0)=h'(x,0) \qquad (33)$$

Equations $(31a)-(32c)$ represent the complementarity problem respectively for stand P and stand P whereas equation (33) says that each forest value at age zero must equal the land value by Proposition 4. As specified by equations $(31a)-(32c)$, and equation (33), the problem of valuing simultaneously both stands can then be solved numerically. The value-matching and smooth-pasting conditions will be used to localize the age cutting boundary. The forest value functions $h(x,a)$ and $h'(x,a)$ are computed iteratively as specified by the following pseudo-code. First, (i) assume that after the $n-1^{th}$ iteration, the land value function is $l^{(n)}$; then (ii) compute forest value function $h^{(n)}$ as solution to $(31a)-(31c)$ and forest value function $h'^{(n)}$ as solution to $(32a)-(32c)$; (iii) deduce the new land value function as $l^{(n+1)}(x)=\max(h^{(n)}(x,0),h'^{(n)}(x,0))$; (iv) continue iteratively until convergence is reached when $h(x,0)$ and $h'(x,0)$ are approximately equal.

The initial value ascribed to the land value can be arbitrarily chosen. We find that the algorithm converges when the land value has initially any positive values. However, the convergence is generally faster when the initial land value function is equal to $\max(c,c'\theta)$ or $\max(c,c'e^x)$. Indeed, this is the minimal land value when both tree species P and P are available.

The numerical discretization to compute $h^{(n)}$ and $h'^{(n)}$ at step (ii) is based on a fully implicit finite difference method. With respect to a fully explicit finite difference method, the implicit method is unconditionally stable and more robust (Brennan and Schwartz 1978). Other numerical methods to solve option valuation problems are discussed in

Wilmott et al (1993). We use the penalty method (Zvan et al, 1998) to solve the linear complementarity problem as in Insley (2002).

Figures 1 and 2 illustrate the cutting age boundary with respectively low and high uncertainty in the case where $\mu < \mu'$. The relative price threshold θ^* for which the decision maker is indifferent between planting any of the two species on a bare land (the switching threshold) corresponds to the relative price for which it is optimal to cut the currently planted species at age zero. It is the same relative price at which the replanting cutting boundary meets the switching one.

Figure 4 shows the land value function in the case where $\mu < \mu'$. The land has a greater value under uncertainty than under certainty and it is more valuable when the uncertainty level, measured by $\overline{\sigma}^2$, increases. Unlike the case under certainty, the land value function has a continuous slope around the relative price switching threshold under uncertainty to prevent any arbitrage. When the relative price tends respectively to zero or to infinity, that is when respectively species P or P is worthless, the reduced land value functions tend toward their single species level c or $c'\theta$ respectively.

5. CONCLUSION

We have examined the decision to undertake projects that differ in that they generate different future income flows. The focus has been extensions of the conventional forestry economics model. In our extended forestry model, two alternative species may be planted, so that outputs as well as timber prices differ across species. This entails more sophisticated planting and harvesting decisions than had been considered before.

When choosing between two alternative species for just one rotation, the decision maker plants immediately if the price of one species is sufficiently high compared to the price of the other species. However, the decision maker prefers to wait if both prices are sufficiently close. This is so even while the decision does not involve any direct cost. Indeed, even in the absence of explicit investment costs, there is the potential cost of establishing the wrong species. This mistake is irreversible because the project involves

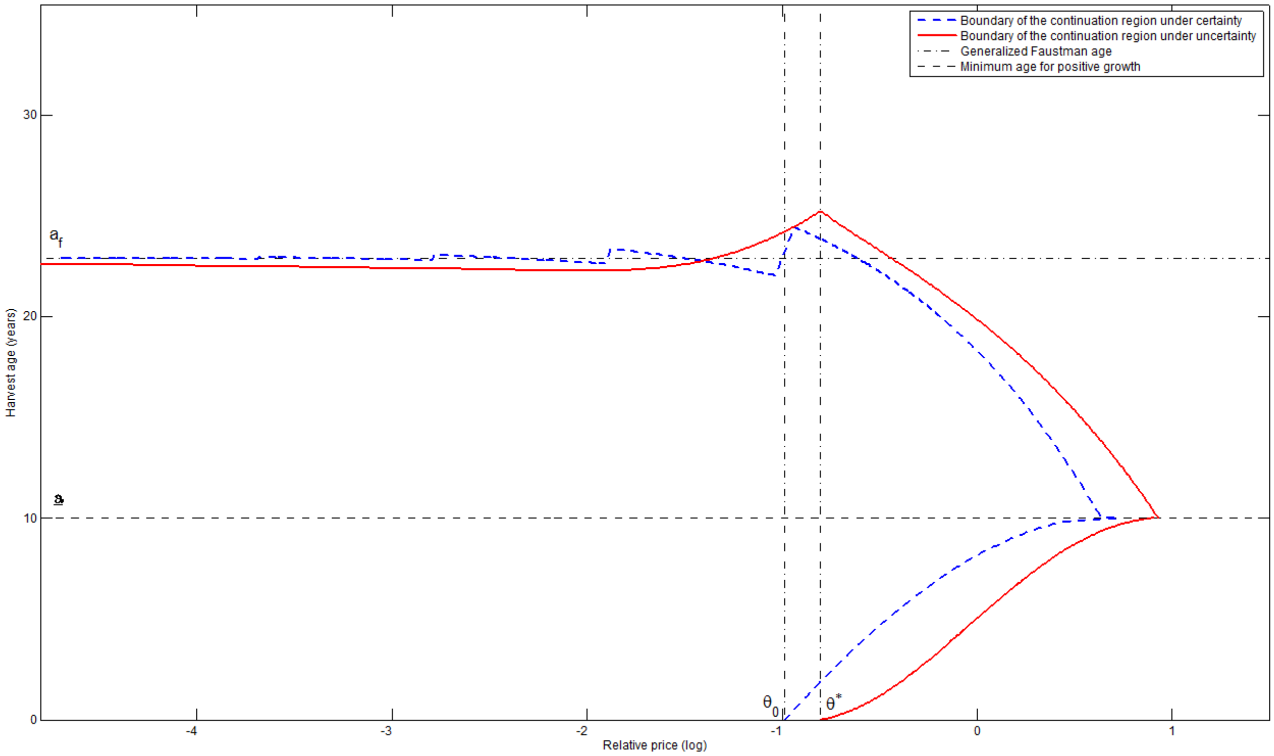

Figure 1: Boundary of the continuation region under low uncertainty

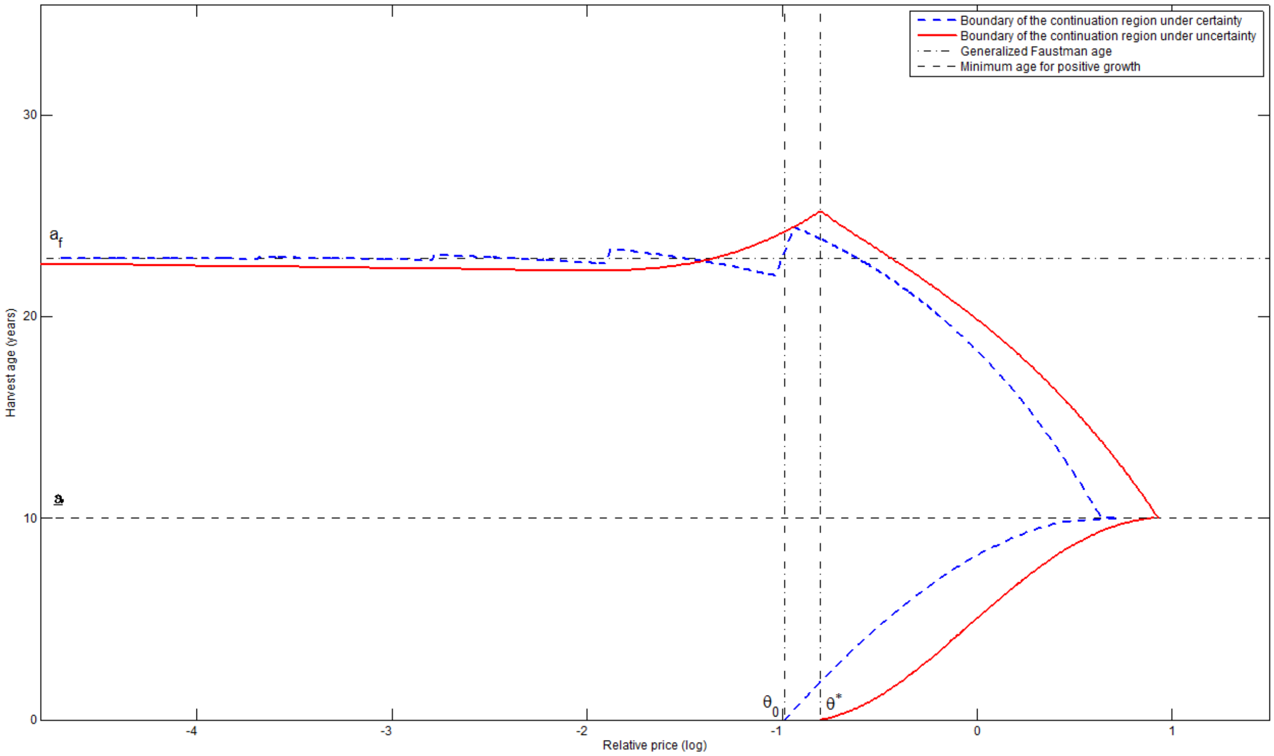

Figure 2: Boundary of the continuation region under high uncertainty

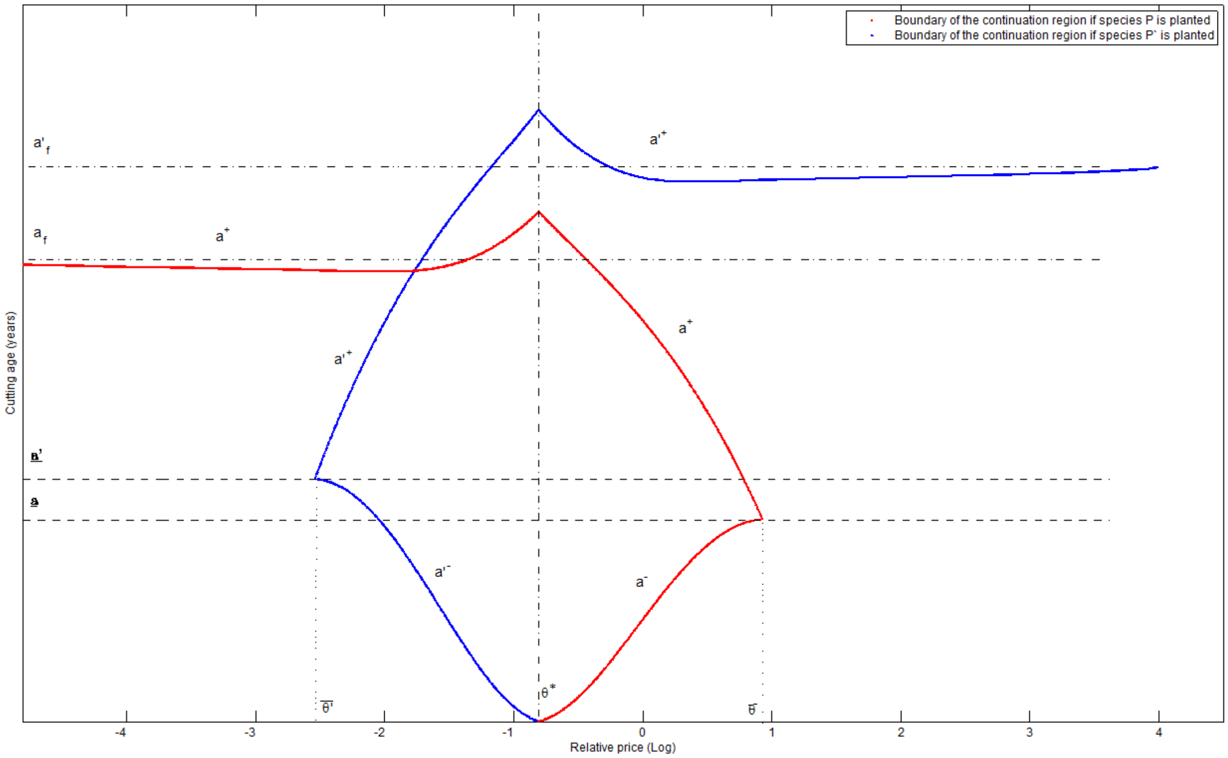

Figure 3: Boundary of the continuation region when species P or the alternative one, P', is planted

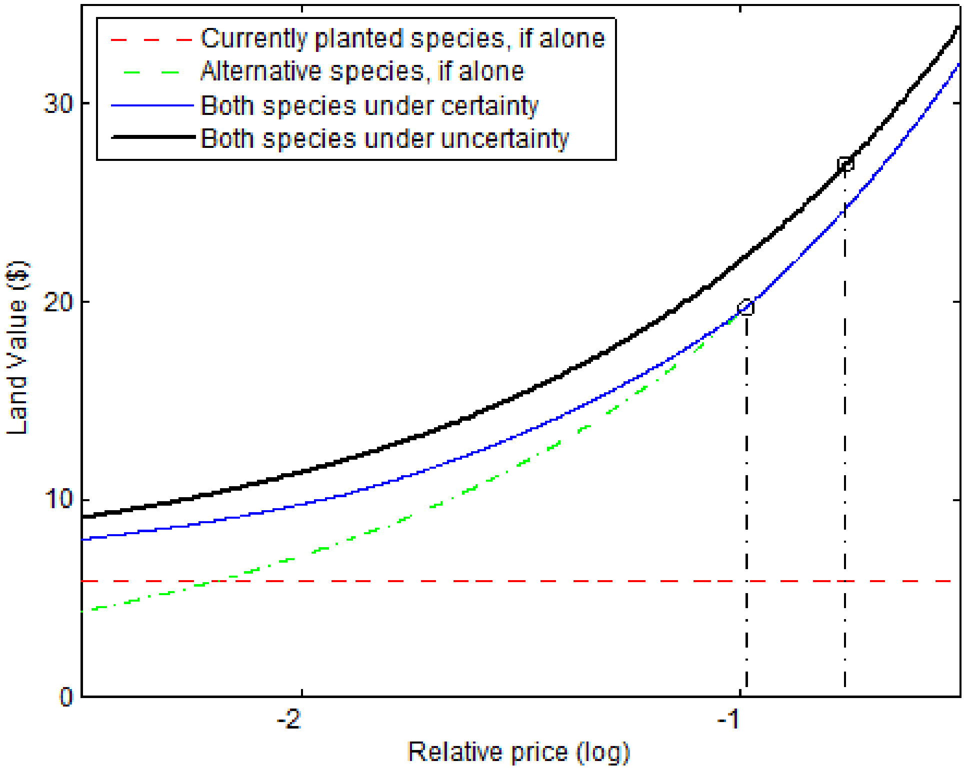

Figure 4: Land value as function of the relative price

one harvest only. Consequently, the decision maker prefers waiting until the prices of the alternative species are sufficiently far away from each other to make the probability of a price reversal acceptably low.

We then considered situations where the decision maker has the opportunity to exploit the forest land for an indefinite number of rotations. Here the decision maker must decide at what age the current stand should be harvested and whether the same species, or the alternative species, should be replanted, immediately or after some delay.

We showed that, absent any planting costs, the investor immediately establishes the species whose relative price exceeds some threshold; otherwise the alternative species is selected. This rule differs from the one rotation case because introducing many rotations has the effect of diluting the irreversibility effect present in the one rotation case. The decision maker plants immediately because it is possible to switch to the alternative species whenever desirable. This is not an easy decision, however, as trees may in that case have to be harvested while they have little or no value. We have shown that, although waiting without planting may help choose the best species, nothing can be gained and some loss may be made, by using that strategy.

We have characterized the value functions and the optimal management strategy in this stochastic repeated rotations context, although they were not provided in explicit form. In the space of relative species prices and wood stand ages, we characterized the set of points where the manager is willing to wait or to exercise the option of harvesting. The exercise frontier divides itself into a zone where the same species is reestablished immediately, and, at higher relative prices a zone where the other species is chosen. The relative price that separates these two zones is independent of stand age. At still higher relative prices, the continuation zone and exercise frontier disappear altogether: a stand of the "wrong" species should then be harvested immediately.

Land value is higher than when one species only is available. It converges to the limiting cases of one species, or the other, when their relative price tends to zero or to infinity. When both species are available, the optimum cutting age is non monotonous but oscillates around Faustman's' age when the relative price is below some threshold value signalling the necessity to switch to the alternative species. This is because the decision maker would rather wait than plant the wrong species and adjust the number of times the same species has to be planted. The land value increases with the uncertainty related to timber prices. It is an increasing function of the relative price with a continuous slope even around the switching threshold to prevent any arbitrage as long as the uncertainty is present. On the contrary, under certainty, the slope of the land value function generally changes at the switching threshold. For relative prices higher than the switching threshold, the optimal harvesting age decreases until it hits the minimum age at which timber volumes become positive. Similar results are established explicitly in the certainty case.

6. APPENDIX

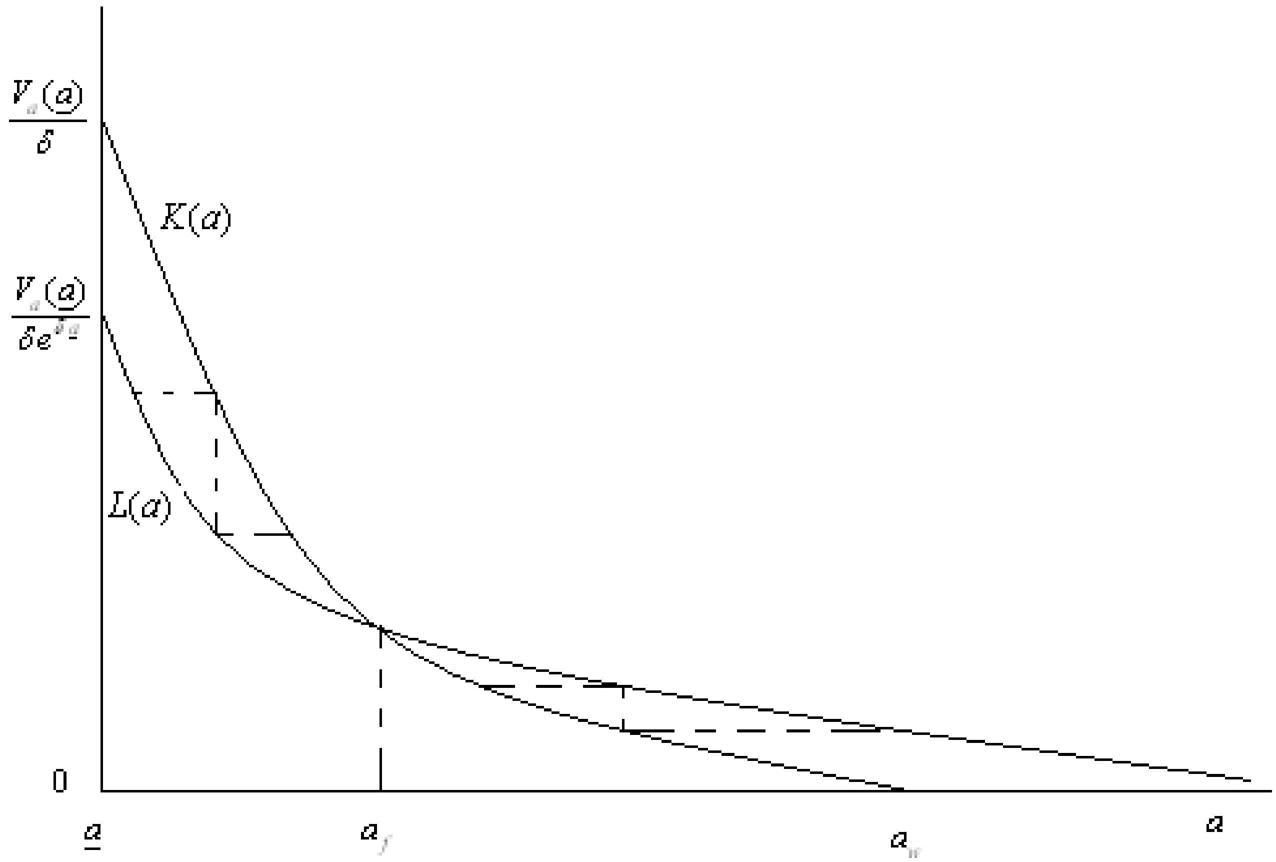

Figure 5: Functions $K(a)$ and $L(a)$.

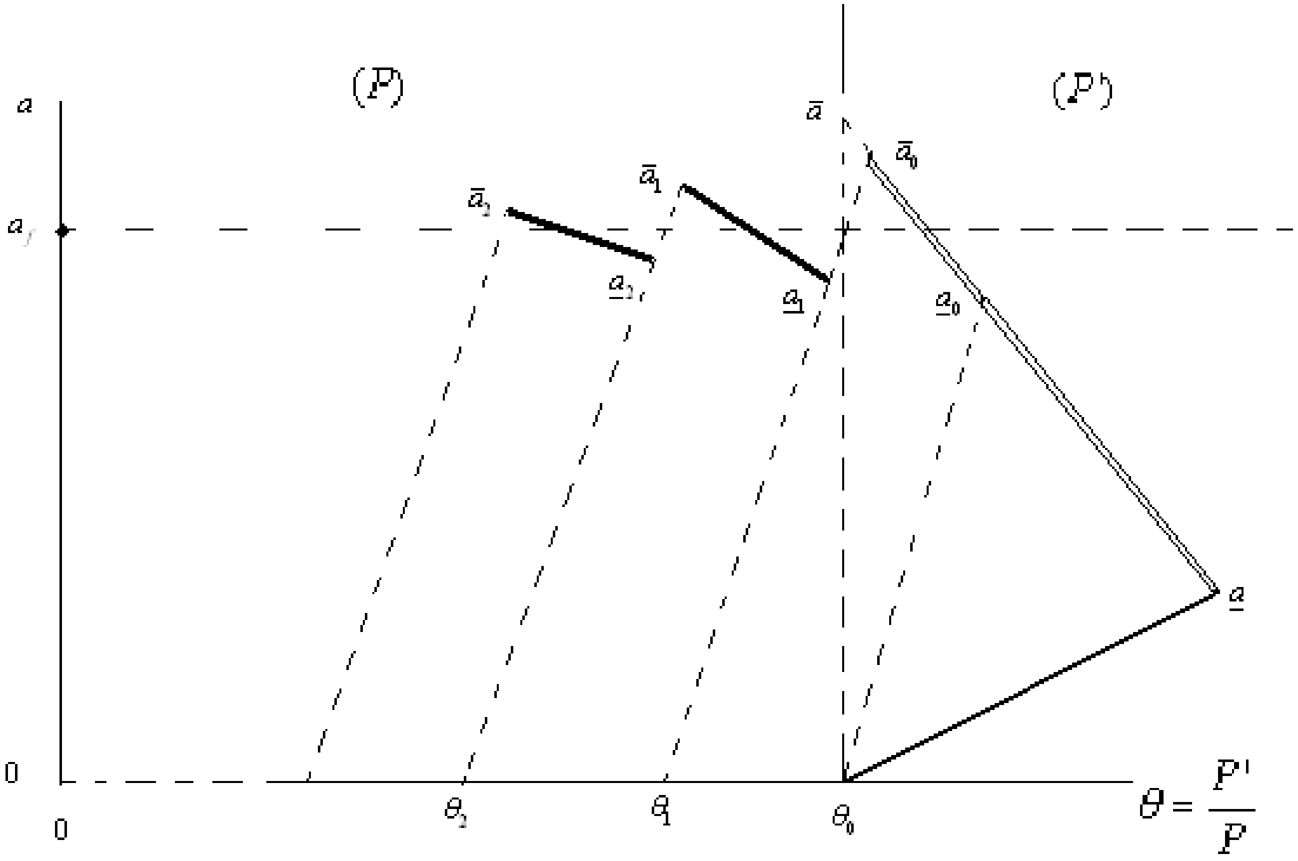

Figure 6: Boundary of the continuation region under certainty

Proof of Proposition 1

Consider a decision maker who chooses the proportion λ of a bare piece of land to be devoted to either one of the species while simultaneously choosing the planting date. The rest of the land $1-\lambda$ is devoted to the other species and planted simultaneously or later. Harvests take place at dates to be chosen optimally for each species. Assuming without loss of generality that species P is planted at T and species P is established at T', the value of the project is then

$$\begin{aligned} F(p,p') \quad &= \quad E \max_{T,T',\lambda\in[0,1]} \Big(\lambda e^{-rT} \max_{s} E_T e^{-rs} V(s) p_{T+s} \\ &\qquad + (1-\lambda)\, e^{-rT'} \max_{s'} E_{T'} e^{-rs'} V'(s') p'_{T+s'}\Big) \end{aligned} \tag{34}$$

where E is the expected value operator conditional on current information, and E_T (respectively $E_{T'}$) is the expected value operator conditional on information at T (respectively T'). Since λ and $1-\lambda$ do not affect the choices of s or s' in problem (34), and since they enter the objective function multiplicatively, they do not affect the choices of T or T'. Consequently λ can be determined given the optimal rules for choosing T and T':

$$\begin{aligned} F(p,p') \quad &= \quad E\left\{\max_{\lambda\in[0,1]}\Big[\lambda \max_{T} E\left(e^{-rT}\max_{s} E_T e^{-rs} V(s) p_{T+s}\right)\right. \\ &\qquad \left. + (1-\lambda)\max_{T'} E\left(e^{-rT'}\max_{s'} E_{T'} e^{-rs'} V'(s') p'_{T+s'}\right)\Big]\right\} \end{aligned}$$

The above problem is linear in λ. Consequently the solution is either $\lambda = 1$ or $\lambda = 0$. This establishes that diversification is not optimum.

Proof of Proposition 1

The only elements not established in the text are the price thresholds and the constants in Formula (7). For a current price level p, sufficiently high relative to p', that is for θ smaller than a critical threshold θ^*, it is optimal to invest immediately in P; then, by (5), $F(p,p') = bp$ or $f(\theta) = b$. On the boundary $\theta = \theta^*$, the value-matching and smooth-pasting conditions (Dixit 1993) imply $f(\theta^*) = b$ and $f_\theta(\theta^*) = 0$. Similarly, at levels p' sufficiently high relatively to p, that is for θ bigger than a critical threshold θ'^*, it is optimal to invest immediately in P; then $F(p,p') = b'p'$ or $f(\theta) = b'\theta$, and the value-matching and smooth-pasting conditions are respectively $f(\theta'^*) = b'\theta'^*$ and $f_\theta(\theta'^*) = b'$. Equations $f(\theta^*) = b$, $f_\theta(\theta'^*) = 0$, $f(\theta'^*) = b'\theta'^*$, and $f_\theta(\theta'^*) = b'$, combined with (7), determine b_1, b_2, θ^* and θ'^*:

$$b_1 \equiv \frac{1-\beta_2}{\beta_1-\beta_2}\left(\frac{[\beta_1/(\beta_1-1)]^{\beta_1}}{[\beta_2/(\beta_2-1)]^{\beta_2}}\right)^{(1-\beta_1)/(\beta_1-\beta_2)} \tag{35a}$$

$$b_2 \equiv \frac{\beta_1-1}{\beta_1-\beta_2}\left(\frac{[\beta_1/(\beta_1-1)]^{\beta_1}}{[\beta_2/(\beta_2-1)]^{\beta_2}}\right)^{(1-\beta_2)/(\beta_1-\beta_2)} \tag{35b}$$

$$\theta^* \equiv \left(\frac{[\beta_1/(\beta_1-1)]^{\beta_1-1}}{[\beta_2/(\beta_2-1)]^{\beta_2-1}}\right)^{1/(\beta_1-\beta_2)} \tag{35c}$$

$$\theta'^* \equiv \left(\frac{[\beta_1/(\beta_1-1)]^{\beta_1}}{[\beta_2/(\beta_2-1)]^{\beta_2}}\right)^{1/(\beta_1-\beta_2)} \tag{35d}$$

Proof of Lemma 1

$F\left(p\right)$ may be written as $F\left(p\right)=\underset{\{a_k\}}{\max}E\sum_{k=1}^{\infty}e^{-r\tau_k}p_{\tau_k}V\left(a_k\right)$ where current time is $t=0$ and $\tau_k=\sum_{s=1}^{k}a_s$ is the sum of all cutting ages from the next harvest to the k^{th} harvest. Since p is a GBM, multiplying p by λ implies that p_s is multiplied by λ for any $s>t$. Then
$F\left(\lambda p\right)=\underset{\{a_k\}}{\max}E\sum_{k=1}^{\infty}e^{-r\tau_k}(\lambda p_{\tau_k})V\left(a_k\right)$
$=\lambda\underset{\{a_k\}}{\max}E\sum_{k=1}^{\infty}e^{-r\tau_k}p_{\tau_k}V\left(a_k\right)=\lambda F\left(p\right)$
Consequently, (9) can be written as
$pF(1)=\underset{s}{\max}Ee^{-rs}\left[p_sV(s)+p_sF(1)\right]$ or $pF(1)=\underset{s}{\max}\left[pe^{-\delta s}V(s)+pe^{-\mu s}F(1)\right]$
$=\ p\underset{s}{\max}\left[e^{-\delta s}V(s)+e^{-\mu s}F(1)\right]$. Thus the maximization which defines the optimal harvest age is independent of the price.

Proof of Lemma 2

First, we will prove that $F\left(p,p'\right)$ is homogenous of degree one in $\left(p,p'\right)$. Define α_k as a dichotomous variable taking the value 1 if species P is planted after the $(k-1)^{th}$ harvest or the value 0 otherwise. Similarly, α'_k is a dichotomous variable taking the value 1 if species P is planted after the $(k-1)^{th}$ harvest or 0 if species P is planted at that time. Thus, α_k and α'_k satisfy $\alpha_k\in\{0,1\}$, $\alpha'_k\in\{0,1\}$, and $\alpha_k\alpha'_k=0$. $F\left(p,p'\right)$ can be written as $F\left(p,p'\right)=\underset{\left\{\alpha_k,\alpha'_k,a_k\right\}}{\max}E\sum_{k=1}^{\infty}e^{-r\tau_k}\left(\alpha_kp_{\tau_k}V\left(a_k\right)+\alpha'_kp'_{\tau_k}V'\left(a_k\right)\right)$. Since p is a GBM, multiplying p_t by λ implies that $p_{t'}$ is multiplied by λ for any $t'>t$. Then $F\left(p,p'\right)$ is homogenous of degree one in $\left(p,p'\right)$. Considering equations $(12a)$ and $(12b)$, the functions $G(p,p',a)$ and $G'(p,p',a)$ are homogenous of degree one in $\left(p,p'\right)$ as well. Now, suppose that species P is currently planted; we want to prove that the optimal cutting age depends on θ_s only, where θ_s is the value of θ when the stand is optimally cut. As $G(p,p',a)=\max_{s\geqslant 0}Ee^{-rs}\left[p_sV(s+a)+F(p_s,p'_s)\right]$, then $G(1,\theta,a) = \max_{s\geqslant 0}\left[e^{-\delta s}V(s+a) + Ee^{-rs}\frac{p_s}{p}F(1,\theta_s)\right]$. Note that $\frac{p_s}{p}$ is the value at time s of a GBM with drift μ and volatility σ whose value is 1 at time 0; thus $\frac{p_s}{p}$ is independent of p (as $\frac{p_s}{p}$ is a GBM with initial value 1). Since it is optimal to cut when

$argmax_s Ee^{-rs}\left[p_sV(s+a) + F(p_s, p'_s)\right] = 0$, the optimal cutting age depends on θ only.

Let $G(p,a) = pg(a)$ where $g(a)$ is a function of the stand age to be determined. Except at harvest age, $G(p,a)$ satisfies Bellman's equation $E\left(dG(p,a)\right) = rG(p,a)da$ where $EdG = G_a da + G_p\mu p da + \frac{\sigma^2}{2}p^2G_{pp}da + o(da)$, $G_a = pg_a(a)$, $G_p = g(a)$, and $G_{pp} = 0$. Consequently $EdG = pg_a(a) + \mu g(a)pda + o(da)$, and Bellman's equation implies that $g(a)$ must satisfy $\delta g(a) = g_a(a)$ for $a \in [0, a_f]$, where $\delta \equiv r - \mu$ and a_f is the optimal harvest age, the same for all harvests by Lemma 1. At any harvest, $G(p,a)$ should satisfy the value-matching and smooth-pasting conditions linking $G(p,a_f)$ and $V(a_f)p + F(p)$. Dividing both conditions by p yields $g(a_f) = V(a_f) + f(\theta)$ and $g_a(a_f) = V_a(a_f)$.

Solving in the usual fashion, one finds that the forest and land value functions, together with c, a constant, and the optimal harvest age are determined by the equations in the Lemma.

As $g(\theta,a) = \frac{1}{p}G(p,p',a) = \frac{1}{p}\max_s Ee^{-rs}\left[p_sV(s+a) + F(p_s,p'_s)\right]$ or $g(\theta,a) = \max_s Ee^{-rs}\left[\frac{p_s}{p}V(s+a) + \frac{p_s}{p}\right.$

that is $g(\theta,a) = \max_s\left[e^{-rs}\frac{Ep_s}{p}V(s+a) + e^{-rs}E\frac{p_s}{p}f(\theta_s)\right]$. While p and θ are possibly correlated, $\frac{p_s}{p}$ is a GBM whose initial value is unity (since by notational definition, p_0 is noted p). Hence $\frac{p_s}{p}$ is independent of θ; as a result it can be replaced by its expected value in $E\left(\frac{p_s}{p}f(\theta_s)\right)$, giving (14$a$). A similar proof gives (14b) and (14c).

Proof Equation (19)

$G(p,p',a) = pg(\theta,a)$ must satisfy Bellman's equation $EdG(p,p',a) = rG(p,p',a)da$.

As $dG = G_a da + G_p dp + \frac{1}{2}G_{pp}dp^2 + G_{p'}dp' + \frac{1}{2}G_{p'p'}dp'^2 + G_{p'p}dpdp' + o(da)$ where $G_a(p,p',a) = pg_a(\theta,a)$,

$G_p(p,p',a) = g(\theta,a) - \theta g_\theta(\theta,a)$,

$G_{pp}(p,p',a) = \frac{\theta^2}{p}g_{\theta\theta}(\theta,a)$,

$G_{p'}(p,p',a) = g_\theta(\theta,a)$,

$G_{p'p'}(a,p,p') = \frac{1}{p}g_{\theta\theta}(\theta,a)$, and

$G_{pp'}(p,p',a) = -\frac{\theta}{p}g_{\theta\theta}(\theta,a)$.

Then $EdG(p,p',a)$

$= pg_a da + \mu pg da + p\theta g_\theta\left(\mu' - \mu\right)da + \theta^2 g_{\theta\theta}\left(\frac{\sigma^2}{2} - \rho\sigma\sigma' + \frac{\sigma'^2}{2}\right)da + o(da)$

and so $\frac{\overline{\sigma}^2}{2}\theta^2 g_{\theta\theta} + \overline{\mu}\theta g_\theta - \delta g + g_a = 0$ where $\overline{\sigma}^2 = \sigma^2 - 2\rho\sigma\sigma' + \sigma'^2$ and $\overline{\mu} = \mu' - \mu$.

Note that θ is a GBM with drift $\mu' - \mu + \sigma^2 - \rho\sigma\sigma'$ and volatility $\overline{\sigma}^2$.

Proof of the Second Order Condition

We will show that the Hessian matrix $\left[\frac{\partial^2 W}{\partial a_i \partial a_j}\right]_{1\leq i,j\leq n+1}$ associated to $W(\theta, a_n, ..., a_0)$ twice continuously differentiable with respect to $(a_n, ..., a_0)$ is negative definite on $]\underline{a}, +\infty[^{n+1}$, where

$$g(\theta,0) = \max_{a_n,\dots,a_0} W(\theta, a_n, ..., a_0)$$

To do so, we will show that its leading principal minors of order k has the sign of $(-1)^k$ for $k = 1, ..., n+1$. First determine $W_{kk} = \frac{\partial^2 W}{\partial a_k \partial a_k}$ for $1 \leq k \leq n+1$, and $W_{kl} = \frac{\partial^2 W}{\partial a_l \partial a_k}$ for l such that $k < l \leq n+1$ as the Hessian matrix is symmetric:

$W_{kl} = -\delta V_a(a_k) e^{-\delta(a_k+...+a_n)} + \delta^2 \sum_{i=0}^{k} V(a_i) e^{-\delta(a_i+...+a_n)} + \delta'^2 c'\theta e^{-\delta'(a_0+...+a_n)}$

$W_{kk} = V_{aa}(a_k) e^{-\delta(a_k+...+a_n)} - \delta V_a(a_k) e^{-\delta(a_k+...+a_n)} - \delta V_a(a_k) e^{-\delta(a_k+...+a_n)}$

$+\delta^2 \sum_{i=0}^{k} V(a_i) e^{-\delta(a_i+...+a_n)} + \delta'^2 c'\theta e^{-\delta'(a_0+...+a_n)}$. Using the first order conditions, we show that W_{kl} is independent of k and l, and that $W_{kl} < 0$: $W_{kl} = -\delta V_a(a_k) e^{-\delta(a_k+...+a_n)} + \delta^2 \sum_{i=0}^{k} V(a_i) e^{-\delta(a_i+...+a_n)}$

$+\delta' \left[V_a(a_k) e^{-\delta(a_k+...+a_n)} - \delta \sum_{i=0}^{k} V(a_i) e^{-\delta(a_i+...+a_n)} \right]$

$W_{kl} = \overline{\mu} V_a(a_k) e^{-\delta(a_k+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k} V(a_i) e^{-\delta(a_i+...+a_n)}$

$W_{kl} = \delta\overline{\mu} K(a_k) e^{-\delta(a_k+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k-1} V(a_i) e^{-\delta(a_i+...+a_n)}$

$W_{kl} = \delta\overline{\mu} L(a_{k-1}) e^{-\delta(a_k+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k-1} V(a_i) e^{-\delta(a_i+...+a_n)}$

$W_{kl} = \overline{\mu} V_a(a_{k-1}) e^{-\delta(a_{k-1}+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k-1} V(a_i) e^{-\delta(a_i+...+a_n)}$

$W_{kl} = \delta\overline{\mu} K(a_{k-1}) e^{-\delta(a_{k-1}+...+a_n)} - \delta\overline{\mu} \sum_{i=0}^{k-2} V(a_i) e^{-\delta(a_i+...+a_n)}$

We continue so and show that,

$W_{kl} = \delta\overline{\mu} K(a_2) e^{-\delta(a_2+...+a_n)} - \delta\overline{\mu} \left[V(a_0) e^{-\delta(a_0+...+a_n)} + V(a_1) e^{-\delta(a_1+...+a_n)} \right]$

$W_{kl} = \delta\overline{\mu} L(a_1) e^{-\delta(a_2+...+a_n)} - \delta\overline{\mu} \left[V(a_0) e^{-\delta(a_0+...+a_n)} + V(a_1) e^{-\delta(a_1+...+a_n)} \right]$

$W_{kl} = \overline{\mu} V_a(a_1) e^{-\delta(a_1+...+a_n)} - \delta\overline{\mu} \left[V(a_0) e^{-\delta(a_0+...+a_n)} + V(a_1) e^{-\delta(a_1+...+a_n)} \right]$

$W_{kl} = \delta\overline{\mu} K(a_1) e^{-\delta(a_1+...+a_n)} - \delta\overline{\mu} V(a_0) e^{-\delta(a_0+...+a_n)}$

$W_{kl} = (\delta' - \delta) V_a(a_0) e^{-\delta(a_0+...+a_n)} - \delta\overline{\mu} V(a_0) e^{-\delta(a_0+...+a_n)}$

$W_{kl} = \delta\overline{\mu} e^{-\delta(a_0+...+a_n)} \left[\frac{V_a(a_0)}{\delta} - V(a_0) \right]$

$W_{kl} = \delta\overline{\mu} e^{-\delta(a_0+...+a_n)} K(a_0) < 0$ as $\overline{\mu} = \delta' - \delta < 0$ and $K(a_0) > 0$.

Denote $W_{kl} = \beta < 0$ and $\alpha_k = \delta e^{-\delta(a_k+...+a_n)} K_a(a_k) < 0$ then $W_{kk} = \alpha_k + \beta$. The k^{th} leading principal minor for $k = 1, ..., n+1$, is therefore

$$\begin{vmatrix} \alpha_1+\beta & \beta & ... & \beta & \beta \\ \beta & \alpha_2+\beta & ... & . & . \\ . & . & ... & . & . \\ . & . & ... & \beta & . \\ . & . & ... & \alpha_k+\beta & \beta \\ \beta & \beta & ... & \beta & \alpha_k+\beta \end{vmatrix}$$

that is denoted $H(\alpha_1, ..., \alpha_k, \beta)$. By subtracting the second line from the first one and then develop the determinant according the first line, we obtain

$$H(\alpha_1, ..., \alpha_k, \beta) = \begin{vmatrix} \alpha_1 & -\alpha_2 & ... & 0 & 0 \\ \beta & \alpha_2+\beta & ... & \beta & \beta \\ . & \beta & ... & . & . \\ . & . & ... & \beta & . \\ . & . & ... & \alpha_{k-1}+\beta & \beta \\ \beta & \beta & ... & \beta & \alpha_k+\beta \end{vmatrix}$$

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \alpha_2 \begin{vmatrix} \beta & \beta & ... & . & \beta \\ \beta & \alpha_3+\beta & ... & . & . \\ . & \beta & ... & . & . \\ . & . & ... & \alpha_{k-1}+\beta & \beta \\ \beta & \beta & ... & \beta & \alpha_k+\beta \end{vmatrix}$$

By subtracting the last line from the first one to compute the last determinant and continue to do so, we obtain

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \alpha_2 \begin{vmatrix} 0 & -\alpha_3 & 0 & 0 & 0 \\ \beta & \alpha_3+\beta & ... & \beta & \beta \\ \beta & \beta & ... & . & . \\ . & . & ... & \alpha_{k-1}+\beta & \beta \\ \beta & . & ... & \beta & \alpha_k+\beta \end{vmatrix}$$

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \alpha_2\alpha_3 \begin{vmatrix} \beta & \beta & ... & \beta \\ . & \alpha_3+\beta & ... & . \\ . & . & ... & . \\ \beta & \beta & ... & \alpha_k+\beta \end{vmatrix}$$

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \alpha_2\alpha_3...\alpha_{k-1} \begin{vmatrix} \beta & \beta \\ \beta & \alpha_k+\beta \end{vmatrix}$$

$$= \alpha_1 H(\alpha_2, ..., \alpha_k, \beta) + \beta\alpha_2\alpha_3...\alpha_k.$$

Now, it is possible to show recursively that the k^{th} leading principal minor has the sign of $(-1)^k$. Indeed, one can check that $H(\alpha_1, \beta) = \alpha_1 + \beta < 0$, and that $H(\alpha_1, \alpha_2, \beta) = (\alpha_1 + \beta)(\alpha_2 + \beta) - \beta^2 = \alpha_1\alpha_2 + (\alpha_1 + \alpha_2)\,\beta > 0$. Assume now that the leading principal minor of order $k-1$, $2 \leqslant k \leqslant n+1$, has the sign of $(-1)^{k-1}$, then $H(\alpha_2, ..., \alpha_k, \beta)$ has the sign of $(-1)^{k-1}$. Consequently, in the expression above, $\alpha_1 H(\alpha_2, ..., \alpha_k, \beta)$ and $\beta\alpha_2\alpha_3...\alpha_k$ have both the sign of $(-1)^k$, therefore the k^{th} leading principal minor $H(\alpha_1, ..., \alpha_k, \beta)$ has the sign of $(-1)^k$.

References

Abildtrup, J. and Strange, N. (1999) “The option value of non-contaminated forest watershed”, *Forest Policy Economics*, (1): 115-25.

Brazee, R.J. and Mendelsohn, R. (1988) “Timber harvesting with fluctuating prices”, *Forest Science*, (34): 359-72.

Brennan, M. and E. Schwartz (1978) “Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis”, *Journal of Finance and Quantitative Analysis*, 13(3): 461-74.

Clarke, H. R., and W.J. Reed (1989) “The tree-cutting problem in a stochastic environment. The case of age-dependent growth”, *Journal of Economic Dynamic and Control*, (13): 569-95.

Conrad, J.M., (1997) “On the option value of old-growth forest”, *Ecological Economics,* (22): 97-102.

Decamps, J.P., T. Mariotti, and S. Villeneuve (2006) “Irreversible Investment in Alternative Projects”, *Economic Theory,* 28(2): 425-48.

Dixit, A.K. (1993) *The Art of Smooth Pasting, Harwood academic publishers.* London, Paris.

Faustman, M. (1849) “On the determination of the value which forest land and immature stands possess for forestry”, reprinted in: M. Gane, ed.. *Martin Faustman and the evolution of discounted cash flow* (1968), Oxford Institute Paper 42.

Forboseh, P.F., Brazee, R.J., and Pikens, J.B. (1996) “A strategy for multiproduct stand management with uncertain future prices”, *Forest Science,* (42): 58-66.

Haight, R.G. (1990) “Feedback thinning policies for uneven-aged stand management with stochastic prices”, *Forest Science,* (36): 1015-31.

Insley, M. (2002) “A Real Options Approach to the Valuation of a Forestry Investment”, *Journal of Environmental Economics and Management,* (44): 471-92.

Jacobsen, J.B., and Thorsen, B.J. (2003) “A Danish example of optimal thinning strategies in mixed-species forest under changing growth conditions caused by climate change”, *Forest Ecology and Management,* (180): 375–88.

Jacobsen, B.J. (2007) “The regeneration decision: a sequential two-option approach”, *Canadian Journal of Forest Research,* (37):439-48.

Kassar, I. and P. Lasserre (2004) “Species preservation and biodiversity value: a real options approach”, *Journal of Environmental Economics and Management,* 48(2): 857-79.

Morck, R., Schwartz, E., and Stangeland, D. (1989) “The valuation of forestry resources under stochastic prices and inventories”, *Journal of Financial and Quantitative Analysis,* 24(4):473-87.

Newman, D. H., C.B. Gilbert, and W.F. Hyde (1985) “The optimal forest rotation with evolving prices”, *Land Economics*, 6(1): 347-53.

Platinga, A.J. (1998) “The optimal timber rotation: an option value approach”, *Forest Science*, (44): 192-202.

Reed, W.J. (1993) “The decision to conserve or harfest old-growth forest”, *Ecological Economics*, (8):45-69.

Saphores, J.D. (2003) “Harvesting a renewable resource under uncertainty”, *Journal of Economic Dynamics & Control,* (28): 509-29.

Thomson, T. (1992) “Optimal Forest Rotation When Stumpage Prices Follow a Diffusion Process”, *Land Economics*, 68(3): 329-42.

Thorsen, B.J. (1999) “Afforestation as a real option. Some policy implications”, *Forestry Science,* (45):171-78.

Thorsen, B.J. and Malchow-Møller, N. (2003) “Afforestation as a real option: Choosing among options”. In: Helles, F., Strange, N., Wichmann, L. (eds.). *Recent Accomplishments in Applied Forest Economics Research.* Kluwer Academic Publishers, Dordrecht, pp. 73-80.

Willassen, Y. (1998) “The stochastic rotation problem: A generalization of Faustmann’s formula to stochastic forest growth”, *Journal of Economic Dynamics and Control,* (22): 573-96.

Wilmott, P., S. Howison, and J. Dewynne (1993) *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press, Oxford.

Zvan, R, P. A. Forsyth, and K. R. Vetzal (1998) “Penalty methods for American options with stochastic volatility”, *Journal of Computational and Applied Mathematics*, (91): 199-218.