A Symmetrical Binomial Lattice Approach, for Modeling Generic One Factor Markov Processes

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Abstract

In this paper we propose a Symmetrical Binomial Lattice Approach that is equivalent to the well-known and widely utilized Lattice of Cox, Ross & Rubinstein (1979) when modeling Geometric Brownian Motion type of processes, but can be utilized for a wide variety of other Markov style stochastic processes. This is due to the highly intuitive construction in which first the expected value expression of the process is directly used and the variance is modeled in a symmetrical lattice, which is added to the first. We then demonstrate its applicability with several Real Options examples, comparing to the Cox model.

Keywords

Symmetrical Binomial Lattice, Markov Processes, Discrete Real Option Modeling.

1 Introduction: Binomial Approximation for Markov Processes

The mathematical complexity associated with derivatives and real options theory derives from the need for a probabilistic solution for the optimal investment decision throughout the life to an option. The solution to this dynamic optimization problem, as described by Dixit and Pindyck (1994), is to model the uncertainty of the underlying asset as a stochastic process where the optimum decision value of investment is obtained by solving a differential equation with the appropriate boundary conditions. In many cases, however, this differential equation has no analytical solution or the simplified assumptions concerning the boundary conditions do not reflect the actual complexity of the problem. In these cases, a discrete approximation to the underlying stochastic process can be used in order to obtain a solution that is computationally efficient for the dynamic valuation problem at hand.

One of these alternatives is the binomial lattice, which is a robust, precise and intuitively appealing tool for option valuation models. The discrete recombinant binomial model developed by Cox, Ross and Rubinstein (1979) to evaluate derivatives is widely accepted as an efficient approximation to the Black, Scholes and Merton's (1973) model due to its ease of use, flexibility and the fact that it converges weakly to a Geometric Brownian Motion (GBM) as the time step (Δt) decreases. Furthermore, as opposed to the Black, Scholes and Merton model, this approach provides the solution to the early exercise of American type options. The approach used by Cox, et al (1979), where the branch nodes recombine due to the fact that the upward movement (u) is the inverse of downward movement (d), means that at each step N, one obtains $N + 1$ node, and not 2^N as in the case of a non-recombining tree. The recombinant lattice is simple and practical to implement in spreadsheet such as Excel or even in decision tree programs. In the approach developed by Brandão, Hahn, and Dyer (2005), for example, the payoffs in each branch correspond to cash flows of each state of the underlying asset.

Often, however, the uncertainty to be modeled does not behave as a GBM type of stochastic diffusion process. This occurs when the value of a variable is a function of a long-term equilibrium level or mean, as is usually the case of non-financial commodities or interest rates. Several authors, such as Bessimbinder, Coughenour, Schwartz (1997, 1988), Laughton and Jacoby (1993) among others, suggest that this type of variable often exhibits auto-regressive behavior and point to the fact that modeling such variable with a GBM can exaggerate the range of values depicted and, as a result, overstate the value of options written on the variable.

But the Cox, et al. (1979) approach only applies to uncertainties that can be modeled through a GBM process. Alternatively Nelson and Ramaswamy (1990) propose a generic binomial model approach that can be used to accommodate other processes than the GBM. Hahn and Dyer (2008) adapt it to model a mean reversion process through a versatile enough methodology that can even be used to model bi-variate problems.

In this paper we develop an alternative Symmetrical Binomial Lattice approach that is equivalent to the Cox, et al. (1979) approach for GBM modeling, but is also more generic and able to model other Markov processes, being more intuitive than the Nelson and Ramaswamy (1990) approach.

2 Cox, Ross & Rubinstein Binomial Model

The binomial approach developed by Cox, Ross & Rubinstein (1979) converges weakly to a Geometric Brownian Motion – GBM - type of Markov diffusion process. This is done by matching the first (Expected Mean) and Second (Variance) moments of the binomial step model with those of the GBM, which is defined by the differential equation: $dS = \mu S dt + \sigma S dz$, where: S is the value or price of the uncertain variable, μ is the drift or growth rate of the stochastic process and σ its volatility parameter. The expected value expression for this process is:

$$E[S_t] = S_0 e^{\mu(t-t_0)}$$

Or considering: $x_t = ln(S_t)$, and: $\Delta t = t - t_0$

$$E[x_t] = x_{t-1} + (\mu - \sigma^2/2)\Delta t,$$ and:

$$Var[x_t] = \sigma^2 \Delta t$$

If we consider the binomial step of length Δt,

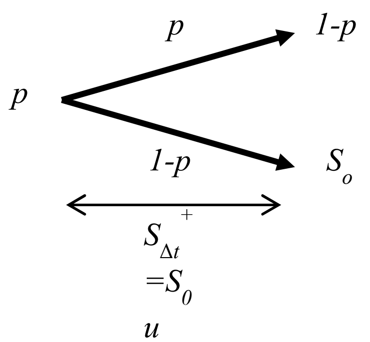

Figure 1 - Geometric binomial step

Where: u and d are respectively the *up* and *down* multipliers of the lattice, and p the probability of an *up* move along this. In order to match the first and second moments of the GBM model, Cox et al. (1979) use these values:

$u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}} = 1/u$, and:

$p = \frac{1}{2}\left(1 + \frac{(\mu-\sigma^2/2)}{\sigma}\sqrt{\Delta t}\right)$, or the more usual expression: $p = \frac{1+\mu-d}{u-d}$

The resulting expected value at t is: $E[S_t] = S_0 * u * p + S_0 * d * (1-p)$, which is dependent on u (and d) and on p, which in turn is a function of μ and σ.

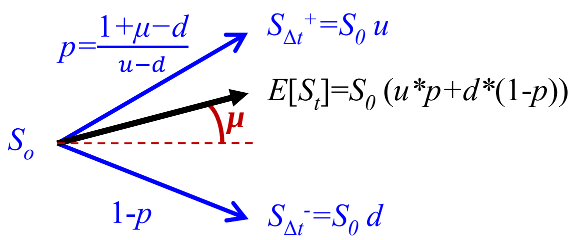

Figure 2 - Cox e al (1979) binomial step

As shown, the drift parameter of the GBM modeled with such a lattice is present only in the probability of the *up* (and complementary *down*) move. The lattice values (estimated with u and d) only model partly the volatility of the process, which is also dependent on the values of p. For this reason when estimating derivatives values, the risk neutral approach of the process is done through the adjustment of these probabilities.

3 Symmetrical Binomial Lattice Approach Proposition

The proposition of this paper is to present an alternate approach for binomial lattice construction that is equivalent to that of Cox. et al. (1979) when modeling GBM yet applicable to a more wide range of stochastic processes and also valid for derivatives and real option calculation.

The basic principle is still to closely match the first and second moment of the process to be modeled, but by using the deterministic expression of the expected value (first moment) directly in the lattice mean value and keeping the lattice up and down movement construction to model the volatility (second moment) of the process.

This can be done in the following steps: consider $x_t = ln(S_t)$, we assume that: $x_t = x^*_t + x'_t$, where: x'_t is the deterministic expected value of the process, so: $x'_t = x'_{t-1} + (\mu - \sigma^2/2)\Delta t$, and: x^*_t are the values of an additive lattice, which models an Arithmetic Brownian Motion with 0 drift, and with U and D as its additive (Up and $Down$) increments.

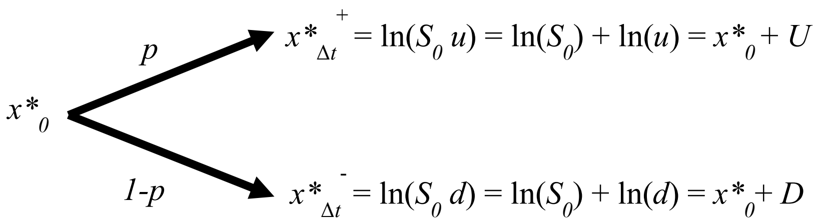

Figure 3 - Additive binomial step

In this case:

$U = \sigma\sqrt{\Delta t}$, $D = -\sigma\sqrt{\Delta t} = -U$, and: $p = \frac{1}{2}\left(1 + \frac{0}{\sigma}\sqrt{\Delta t}\right) = 0.5 = 1 - p$,

To represent the whole process of building the additive step for the x variable, we can use the following diagram:

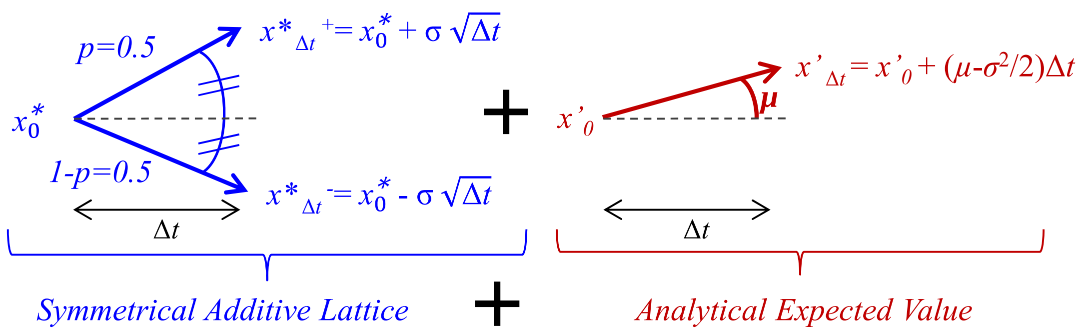

Figure 4 - Symmetrical Lattice construction steps - GBM

In this lattice the branches are symmetrical around the expected value expression.

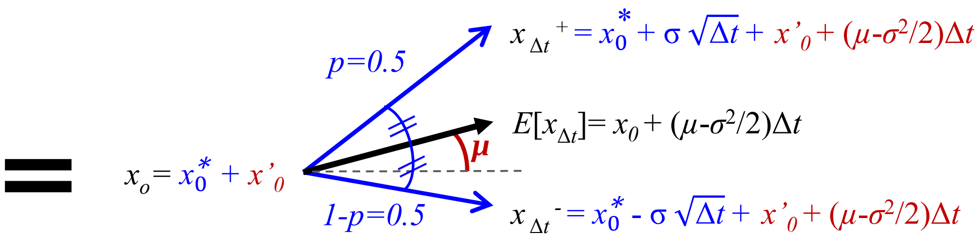

Figure 5 - Symmetrical Lattice Nod - GBM

To obtain the multiplicative symmetrical lattice, it is only necessary to calculate at each nod: $S_t = e^{x_t}$, and use p = 0.5 throughout the whole lattice.

4 Examples of real options calculation with both models

We will now calculate a real option using both Cox, et al. lattice and the Symmetrical lattices approach and show the equivalent result of both methods.

We consider a project with a present quarterly Cash Flow of 10 $million. This Cash Flow has a volatility of σ = 40% and a drift rate (risk adjusted) of μ=8%, both early values. The project´s risk adjusted discount rate is k = 12% and the risk free rate is r = 6%. The project´s Cash Flows are expected to grow rate μ during 5 years (in 20 quarterly periods) after which a perpetuity without grow is used. With the above premises the Present Value of the project is: PV_0 = 456.5 $ million.

To obtain the risk free equivalent Present Value, we use a risk free growth rate μ' for the cash flows estimates as follows: $\mu' = \mu - (k - r) = 2\%$. When discounting at the risk free rate r the Cash Flows with this risk free growth rate, the Present Value of the project is: PV_0' = 454.1 $ million.

We first model a lattice with 20 quarterly periods for the Cash Flows using the Cox, et al. model. In this case, using σ = 40%, $\mu' = 2\%$, and $\Delta t = 0.25$, we obtain:

$u = e^{\sigma\sqrt{\Delta t}} = 1.2214,\ d = e^{-\sigma\sqrt{\Delta t}} = 0.81873$, and $p = \frac{1+\mu'\Delta t - d}{u-d} = 0.4626$

Starting with the end period (20^{th} quarter) and considering perpetuity, we then discount the lattice at the risk free rate r, summing the Cash Flow values of the lattice at each nod, and weighting by the probabilities p and (1-p) above, we end up at time 0 with a Present Value of the project is: PV_0' = 462.5 $ million, or an error of 1.3% over the base case value.

Now we model the alternate Symmetrical Lattice through the process described earlier above.

Again using: σ = 40%, $\mu' = 2\%$, and $\Delta t = 0.25$, we obtain:

$\mu' - \sigma^2/2 = -6\%$ (necessary for x'_t calculation, and already the risk free adjusted drift rate),

$U = \sigma\sqrt{\Delta t} = 0.200$, and $D = -0.200$, with $p = 0.5 = (1-p)$

Again we discount the risk free rate r the Cash Flows of this Symmetrical Lattice, considering perpetuity, from the end period up to the start, always weighting by 0.5, and now we end up at time 0 with a Present Value of the project is: PV_0' = 457.2 $ million, or an error of 0.2% over the base case value.

This result shows a slightly more accurate Symmetrical Lattice than the traditional Cox, et al. model, but this cannot be generalized since it can be the result of the parameters of the example used. But it points to the equivalence of both approaches and the validity of the Symmetrical Lattice for GBM modeling.

Although we are modeling the projects Cash Flows in the lattices, the underlying asset, over which any option should be exercised, is the Present Value of the project itself. We now incorporate in the above project two different Real Options for the project at hand.

First an expansion option available at any time during the 5 year lattice forecasting in quarterly periods, which is modeled as an American Call Option that increases in 90% the Project Value at the point of exercise at a cost of 400 $ million.

Simultaneously we model an abandonment option for the same time span, which is modeled as an American Put Option, in which the project can be abandoned against a fixed value (sale of remaining assets) of 350 $ million.

These options are easily modeled with binomial lattices, and in the case of the lattices described above, as we modeled directly the projects Cash Flows there is no need for altering the construction to incorporate these.

When incorporating the Options above into the lattices we end up at the start of these with the expanded Present Values, that is the traditional Present Values plus the Option Value of any flexibility incorporated in the lattices.

From the Cox, et al. lattice model we get a Real Option value of: 181.4 $ million or a 39.9% increase over the base case Present Value of 456.5 $ million.

Using the Symmetrical Lattice approach with the same Real Options incorporated we get a Real Option value of: 184.9 $ million or a 40.7% increase over the base case.

The above example shows that the Symmetrical Lattice approach is similar to that of Cox, et al. model when modeling GBM type of uncertain variables, and that securities and options written on such a model yield similar results, making thus the Symmetrical Lattice a valid method for options and real options calculation.

Nevertheless the Cox, et al. model is widely used as the main binomial approach for valuing derivatives and options and another model that does the same think would not be a significant contribution unless it brings some sort of new application or improvement over the traditional one.

The main difference of both is that with the Cox, et al. model, the drift rate of the process is not built in the lattice construction but incorporated into the transition probability of each nod. The symmetrical model on the other hand incorporates both the drift and the volatility in the lattice construction separating the two moments of the stochastic process in two separate parts of this construction. The transition probability only weights symmetrically the discounting of the values along the lattice.

Although this approach seems more intuitive for the new lattice practitioner, it still is no substitute for the Cox, et al. model when it comes to modeling GBM uncertainties. Usually the Present Value of future cash flows is assumed to follow a random walk type of process, or a GBM, according to Samuelson theorem (*apud* Copeland & Antikarov, 2003) and this is used to model a decision tree lattice based on the Cox, et al. model. Nevertheless, frequently the uncertainty involved with project cash flows does not follow a GBM as is the case of commodities and other commodity dependent projects. Although according to Samuelson theorem the present value of these cash flows would still have a random walk behavior, the cash flows themselves certainly would not. A number of authors support the general view that commodity dependent cash flows, among others, generally would follow some type of auto regressive behavior. In the examples above we directly modeled the cash flows on the lattices to calculate the flexibility value. In case these would not follow a GBM type of diffusion process, then the Cox et al modeling would not be fit to estimate the real option value present, or might be overestimating it greatly.

In the following section we will adapt our Symmetrical Lattice to model a mean reversion process and show that it can be modified to fit a great number of similar and generic autoregressive processes.

5 Generic Symmetrical Binomial Lattice Model

The use of binomial lattices similar to the classic GBM model of Cox, et al. (1979) to model other Markov processes has been scanty due to the fact that such models often produce transition probabilities greater than 1 or less than zero when the influence of mean reversion is particularly strong. Consequently, discrete trinomial and multi-nomial trees (Hull, 1999) and Monte Carlo simulation models have been the primary methods used to model MR processes. Unfortunately, trinomial trees, such as those suggested by Tseng and Lin (2007), Clewlow and Strickland (1999), Hull and White (1994[a], 1994[b]) and Hull (1999), require more involved methodologies for specifying valid branching probabilities and lattice cell sized to ensure convergence of the stochastic process. This requires more sophisticated programming and results in difficulty in applying trinomial trees to a wide range of specific projects and cases.

Monte Carlos simulation approaches such as the Least Squares method (LS) of Longstaff and Schwartz (2001) are able to accommodate almost any stochastic process, including a combination of various processes, thereby eliminating the so-called “curse of dimensionality and modeling". However, the shortcoming of these models is in modeling decisions, which can pose problems in the modeling of compound options, for example.

5.1 Mean Reversion Lattice

A Mean Reverting (MR) stochastic process is a Markov process in which the direction and intensity of deviation are a function of the long term equilibrium level to which the current price must revert. The logic behind a Mean Reverting Model derives from microeconomics: when prices are depressed (or below their long term equilibrium level), the demand for this product tends to increase while the production tends to decrease. This is due to the fact that consumption of a commodity, for instance, increases as prices decrease, while low returns to producers will lead to the decision to postpone investment or to close less efficient units, thereby reducing the supply of the product. The opposite will occur if prices are high (or above the long term equilibrium level or mean). As an example, empirical studies (Pindyck & Rubinfeld, 1991) have shown that these microeconomic forces do indeed cause oil prices to exhibit mean reverting stochastic behavior.

The simplest form of MR process is the single factor Ornstein-Uhlenbeck process, also called Arithmetic MR process, which is defined by:

$$dx_t = \eta(\bar{x} - x_t)dt + \sigma dz_t$$

where x_t is the natural log of the variable S_t, η the mean reversion speed, $\bar{x}$ is the long term average to which x_t reverts, σ the volatility of process and dz is the standard Wiener process. The natural logarithm of the variable is used since in the case of commodities it is generally assumed that these prices have a lognormal distribution. This is convenient because since $S = e^x$, S cannot be negative. Therefore, the expected value and variance of the Orstein-Uhlenbeck process are given by Dixit and Pindyck (1994), in their discrete model form:

$$E[x_t] = \bar{x} + (x_{t-1} - \bar{x})e^{-\eta\Delta t}$$

$$Var[x_t] = \frac{\sigma^2}{2\eta}(1 - e^{-2\eta t})$$

5.1.1 Nelson and Ramaswamy MR Lattice approach

Nelson and Ramaswamy (1990) proposed an approach that can be used in a wide range of conditions, and which is appropriate for the Ornstein-Uhlenbeck process. Their model is a simple binomial sequence of *n* periods of duration Δt, with a time horizon T: $T = n\,\Delta t$, which then allows a recombinant binomial tree to be built.

Considering the general form for the differential equation of a Markov type stochastic process given by: $dx = \mu(x,t)dt + \sigma(x,t)dz$, the Nelson and Ramaswamy model is given by the following equations:

$x_t^+ \equiv x + \sqrt{\Delta t}\sigma(x,t)$ (*up* movement)

$x_t^- \equiv x - \sqrt{\Delta t}\sigma(x,t)$ (*down* movement)

$p_t \equiv 1/2 + 1/2\sqrt{\Delta t}\dfrac{\alpha(x,t)}{\sigma(x,t)}$ (*up* probability)

$1\text{-}p_t$ (*down* probability)

For the Ornstein Uhlembeck process these equations would be:

$\alpha(x,t) = \eta(\bar{x} - x_t)$, and

$$\sigma(x,t) = \sigma$$

Hahn & Dyer (2008) and Bastian-Pinto, Brandão & Hahn (2009) use this approach to model bivariate lattices, on which at least one of the variables follows a Mean Reversion. However, in this model, the probability pt can assume values or values greater than 1. This condition is remedied by censoring the probabilities pt (and therefore: 1- pt), to the range of 0 to 1 in the following manner:

$$p \equiv \begin{cases} \frac{1}{2} + \frac{1}{2}\frac{\eta(\bar{x}-x)}{\sigma}\sqrt{\Delta t} & \text{if } p \geq 0 \text{ and } p_t \leq 1 \\ 0 & \text{if } p_t < 0 \text{ , } p_t \text{ is censored} \\ 1 & \text{if } p_t > 1 \text{ , } p_t \text{ is censored} \end{cases}$$

5.1.2 Symmetrical Lattice Approach to Mean Reversion

The Nelson & Ramaswamy approach is similar to that of Cox. et al., since the drift of the expected value of the process is regulated by the value of p, which in this case varies with x.

This model can be easily adapted to a Symmetrical model similar to the one described for GBM modeling. We use the approach suggested by Hull and White (1994[a], 1994[b]) as described in Clewlow and Strickland (1999) and in Hull (1999), for the case of a trinomial tree model of a MR process. First, we define a Symmetrical Additive Lattice, which models an Ornstein-Uhlenbeck arithmetic process with a long term mean equal to zero: $\bar{x}^* = 0$, and initial value of zero: $x_0^* = 0$. In this lattice the nodes will have a value of x_t^*. The expected values of the Ornstein-Uhlenbeck x'_t model are added to the value of the nodes in each period using the real long term average of the process: $\bar{x}$, and the real starting value of: x_0. Hence, this Lattice of values x_t is used to obtain the Lattice of a price process S_t with lognormal distribution defined by: $S_t = e^{x_t}$, and $x_t = x'_t + x_t^*$.

Using the same construction as with the Symmetrical GBM Lattice, and the parameters of the Nelson & Ramaswami model, we have the following relationships for the Additive Symmetrical Lattice:

$$x^{*+} = x^* + \sigma\sqrt{\Delta t}$$

$$x^{*-} = x^* - \sigma\sqrt{\Delta t}$$

$$p_{x_t^*} = \frac{1}{2} + \frac{1}{2}\frac{\eta(-x_t^*)}{\sigma}\sqrt{\Delta t}$$

This last value will also need censoring to the [0,1] range since depending on the values of x_t^* the expression of $p_{x_t^*}$ can exceed this range. Therefore we will use the following expression:

$$p_{x_t^*} = max\left(0, min\left(1, \quad \frac{1}{2} + \frac{1}{2}\frac{\eta(-x_t^*)}{\sigma}\sqrt{\Delta t}\right)\right)$$

Bastian-Pinto, Brandão & Hahn (2010) use also a slight variation of this approach that does not need censoring and develop the following expression:

$$p_{x_t^*} = \frac{1}{2} + \frac{1}{2}\frac{\eta(-x_t^*)}{\sqrt{\eta^2(-x_t^*)^2\Delta t + \sigma^2}}\sqrt{\Delta t}$$

But this model returns slightly overestimated values of variance, due to the nature of the Lattice construction. These authors use this approach together with the Symmetrical model here proposed and model a switch real option. Although this non-censored model is interesting and applicable, we use the censored probability approach shown above in this paper due to the apparently higher precision of the model. The x value after i up movements, and j down movements will be: for $t=(i+j)\Delta t$

$$x_{(i,j)}=\bar{x}+\left(x_0-\bar{x}\right)e^{-\eta(i+j)\Delta t}+\underbrace{\left(i-j\right)\sigma\sqrt{\Delta t}}_{x^*}\text{, or:}$$

$$x_{(i,j)}=\bar{x}\left(1-e^{-\eta(i+j)\Delta t}\right)+x_0e^{-\eta(i+j)\Delta t}+\underbrace{\left(i-j\right)\sigma\sqrt{\Delta t}}_{x^*}$$

The Symmetrical Lattice construction approach is shown here:

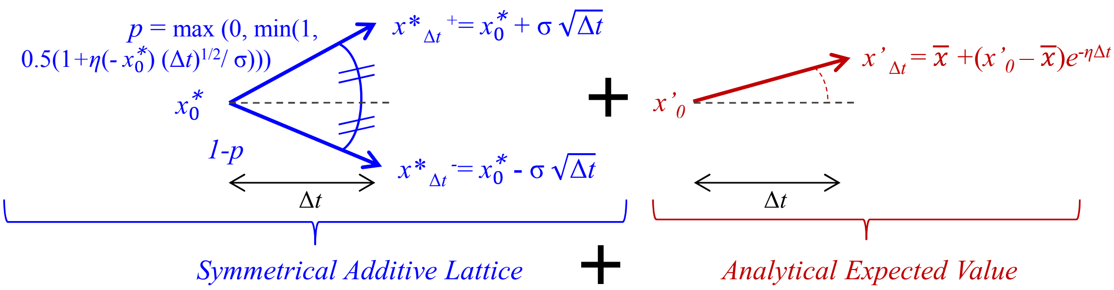

Figure 6 - Symmetrical Lattice construction steps – Ornstein Uhlembeck

Or:

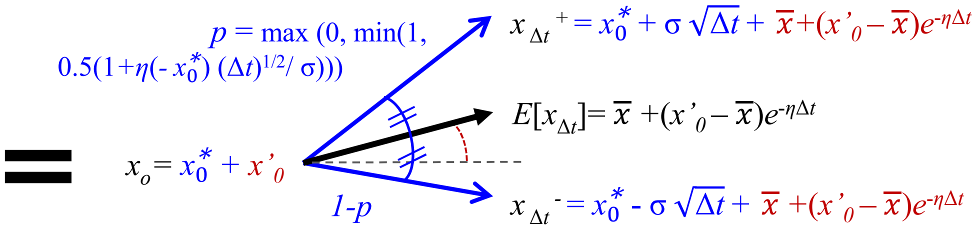

Figure 7 - Symmetrical Lattice nod - Ornstein Uhlembeck

We can see a representation of a 15 period (with Δt = 0.25) combination of the above construction scheme (all in geometric form) in the lattices bellow.

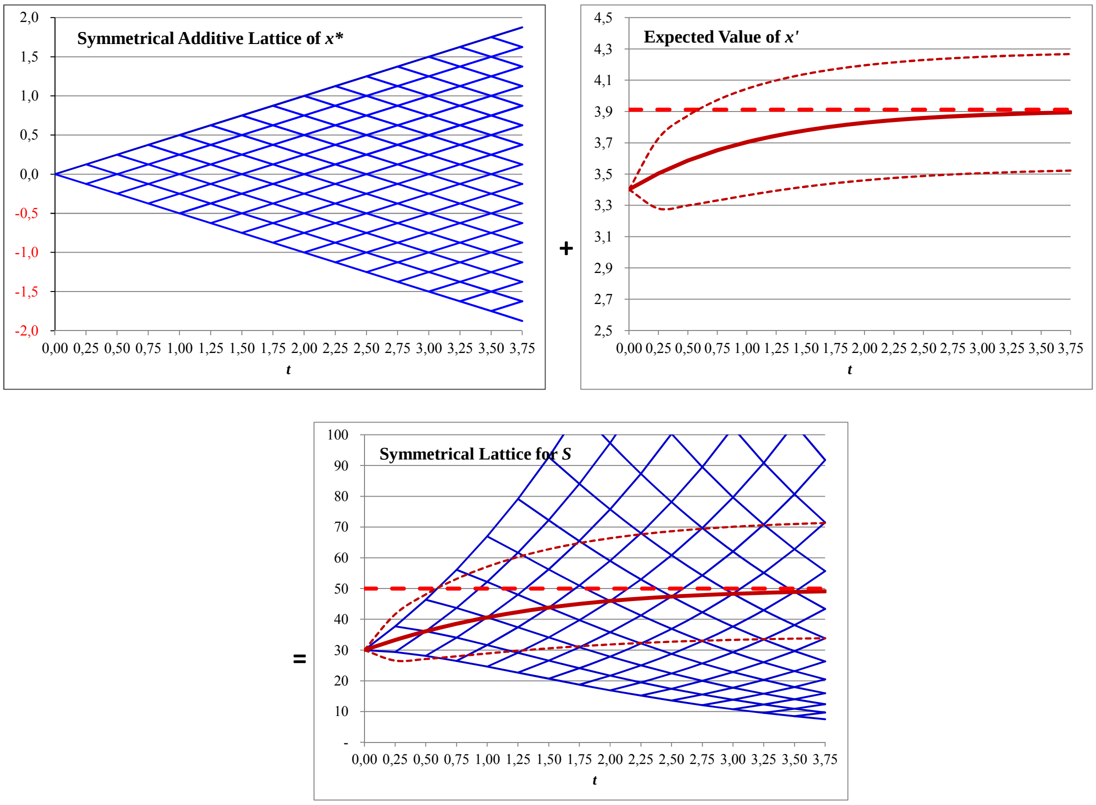

Figure 8 - Combination of Symmetrical Lattice and Expected Value showing up and down 95% cert. levels

Attention must be made to the value of $\overline{x}$: it is not directly the natural logarithm of the equilibrium level (mean) of the price modeled : $\overline{S}$. Applying Ito´s Lema to the differential equation of the geometric mean reversion model:

$$dS = \eta\left(ln(\overline{S}) - ln(S)\right)Sdt + \sigma Sdz$$

We end up with: $\overline{x} = ln(\overline{S}) - \frac{\sigma^2}{2\eta}$

The Symmetrical Lattice for the geometric MR process, defined by: $S_t = e^{x_t}$, is obtained by directly transforming $x_{(i,j)}$ values in $S_{(i,j)}$. We note that in this Symmetrical Lattice, the adjustment for risk neutrality is given in the equation of expected value of the process, altering the value of x_t to:

$$x_{(i,j)} = \left(\overline{x} - \lambda_x/\eta\right)\left(1 - e^{-\eta(i+j)\Delta t}\right) + x_0 e^{-\eta(i+j)\Delta t} + \underbrace{(i-j)\sigma\sqrt{\Delta t}}_{x^*}$$

Where: λx is the risk premium of the variable x, and $\lambda x/\eta$ the corresponding normalized risk premium.

This adjustment to transform a MR process into a risk neutral process is also simpler than that of the censored model, which requires that it be done in the transition probabilities along the whole lattice.

In the following section we will apply the Symmetrical Mean Reversion Lattice to the valuation of a hypothetical real option, similarly to the example with a GBM approach seen previously.

5.2 Example of real option calculation using Symmetrical Mean Reversion Lattice

We now consider a project with the same characteristics as in the GBM example: a present (t=0) quarterly Cash Flow of $S_0 = 10$ \$ million. This Cash Flow has a volatility of σ = 40% and a mean reversion speed parameter of η=1, both early values. The project´s risk adjusted discount rate is k = 12% and the risk free rate is r = 6%. The project´s Cash Flows are expected to assume a mean reversion behavior to a long term equilibrium level of: $\overline{S}$ = 15 $ million for 5 years (in 20 quarterly periods) after which a perpetuity without grow is used. Since we assume that the Cash Flows have a mean reversion behavior, the perpetuity calculation is different from a constant grow model (like a GBM). The perpetuity formula for an Ornstein Uhlembeck process is: $P_t = \frac{CF_t}{k} + \frac{CF_t - \overline{CF}}{k+\eta}$, where CF_t is the last Cash Flow before perpetuity is assumed and $\overline{CF}$ is the value to which CF converges.

With the above premises the Present Value of the project is: PV_0 = 466.5 $ million.

To risk free adjust a Mean Reversion process the normalized risk premium $\lambda x/\eta$, is subtracted from $\overline{x} = ln(\overline{S}) - \frac{\sigma^2}{2\eta}$. We used a normalized risk premium for the Cash Flow process of: $\lambda x/\eta$=0.199 and obtain a value for the risk free adjusted equilibrium level $\overline{x}_{rf}$ =2.403. When discounting at the risk free rate r the Cash Flows with this risk premium, the Present Value of the project is now: PV_0' = 465.5 $ million.

Again we first model a lattice with 20 quarterly periods for the log of the Cash Flows using the Symmetrical additive model. In this case, using σ = 40%, $\eta = 1$, $\overline{x}_{rf}$ =2.403, S_0 = 10 $ million and $\Delta t = 0.25$, we obtain: $U = 0.200$, $D = 0.200$. The transition probability p of an up move along the lattice is calculated using the censoring formula give above.

The Symmetrical Lattice for S without risk free adjustment (e.i. with long term mean of $\overline{S} = 15$ $ million) and already in geometric form is shown here bellow, with indication of accumulated probabilities at each nod, as well as censored nods (probability of occurrence = 0).

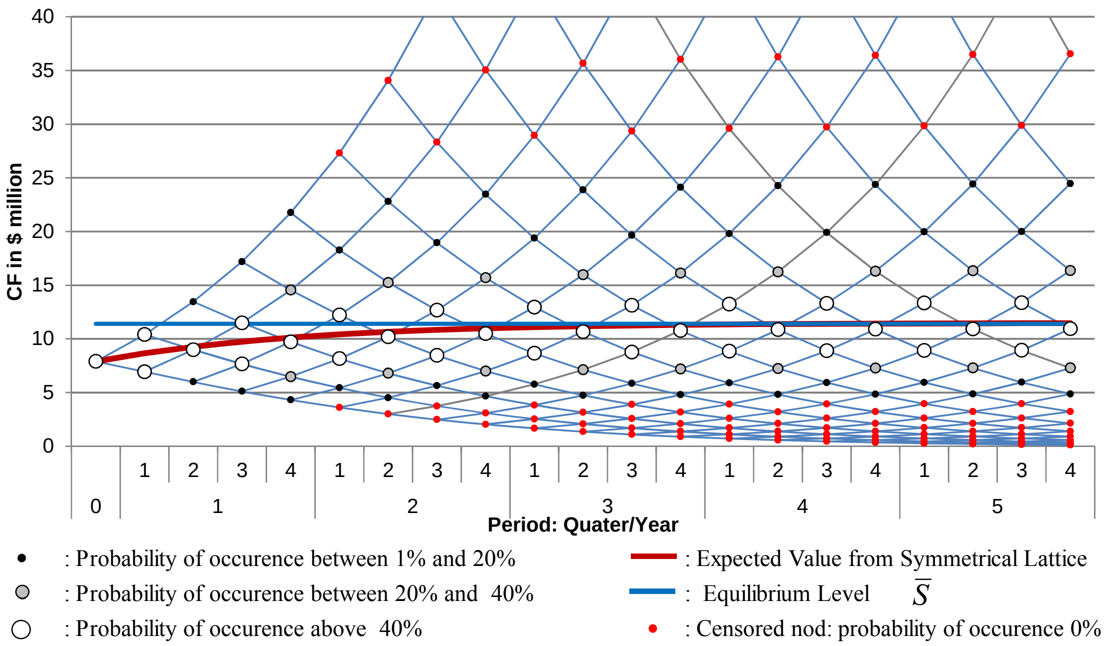

Figure 9 - Symmetrical Lattice for values of Cash Flow (not risk adjusted)

Again starting with the end period (20th quarter) and considering perpetuity (with a mean reversion formula), we then discount the lattice at the risk free rate r, summing the Cash Flow values of the lattice at each nod, and weighting by the probabilities p and (1-p) calculated at each nod and censored when needed , we end up at time 0 with a Present Value of the project is: PV_0' = 470.6 $ million, or an error of 0.9% over the base case value.

Now we incorporate the same real options calculated for GBM model: an expansion possibility of 90% of the project value at a cost of 400 $ million and an abandonment option for a value of 350 $ million.

Using the same backwards discounting procedure as with the GBM model, but considering the varying transition probabilities of the MR lattice, we end up in *t* = 0, with an option value of 29.5 $ million or 6,3% above de base case.

Of interest is to note that the options values with mean reversion modeling is significantly lower that when modeling the Cash Flows with GBM. This is coherent with theory since mean reversion is an inhibiting factor for option exercise by its particular nature. Another point of interest is that although with the GBM process the abandonment option has a significant value on its own, 85.5 $ million with the Symmetrical Lattice, in the MR model its value is close to zero. This is explained by the characteristic of the MR again: if the value of the Cash Flow is significantly low the MR dynamics will correct on average this value inhibiting any abandonment exercise. The same will happen with the expansion option but to a lesser extent.

5.3 Mean Reversion with Drift

As mentioned the binomial approach of Cox et al is limited to GBM type of derivatives modeling. We propose a Symmetrical Lattice that is more generic for other Markov type of stochastic processes modeling. Although Mean Reversion can be modeled through the Nelson & Ramaswani (1990) approach, variations of this stochastic process cannot, or at least would implicate in limitations and accommodations that could turn the model too complex to be practical.

We will now model in a Symmetrical Lattice a Mean Reversion process to which a deterministic drift rate is added to the long term equilibrium level. Pindyck (1999) models commodities prices with a somewhat similar approach using a mean reversion with an equilibrium level that evolves quadradically but in a deterministic way. There are several two factor models in the literature such as Gibson & Schwartz (1990), Schwartz (1997) model 2, and Schwartz & Smith (2000) among others. The model here used can be seen as a particular case of these two factor models, where the equilibrium level does not follow a stochastic process but has a deterministic evolution. Such a model is difficultly modeled through traditional binomial lattices, but can be very easily incorporated to the Symmetrical approach here proposed. We will only need to include the drift component in the expected value calculation step of the lattice construction. This approach is used by Ozorio, Bastian-Pinto & Baidya (2011) and they name this process as Mean Reversion with Drift – MRM-D. Its differential formulation is: $dx_t = -\eta x_t dt + c dt + \sigma dz$, and expected value, using the same formulation as with the other processes : $E_0[S_t] = \exp\left(x_0 e^{-\eta t} + \ln\left(\overline{S}_o\right) + \mu t + \frac{1}{2}(1 - e^{-2\eta t})\frac{\sigma^2}{2\eta} \right)$

Therefore we consider a project with the same characteristics as in the previous examples: a present (*t*=0) quarterly Cash Flow of $S_0 = 10$ $ million. This Cash Flow has a volatility of σ = 40% and a mean reversion speed parameter of η=1, both early values. The project´s risk adjusted discount rate is k = 12% and the risk free rate is r = 6%. The project´s Cash Flows are expected to assume a mean reversion behavior to a long term equilibrium level that is presently: $\overline{S_0}$ = 15 $ million but that is expected to grow at a yearly rate of μ = 5% for 20 quarterly periods. After this period a perpetuity without grow is assumed.

The Symmetrical Lattice for S_t before risk free adjustment and with long term mean starting at $\overline{S_0} = 15$ $ million, and growing at μ = 5% is shown here bellow, with indication of accumulated probabilities at each nod, $\overline{S_t}$ plot, expected value plot and up and down boundaries for 95% certainty of S_t.

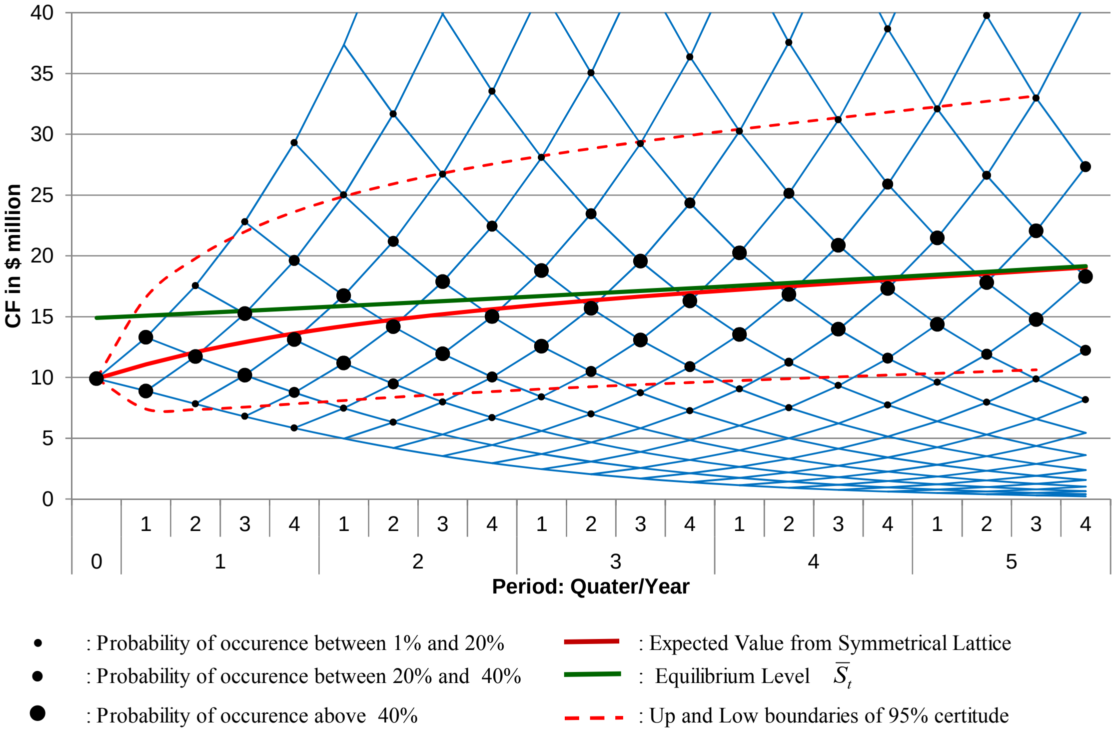

Figure 10 - Symmetrical Lattice for values of Cash Flow (not risk adjusted)

For the risk neutral transformation we can now use the same adjustment as with the MRM process.

Using the same discounting procedure as with the previous examples, the Present Value of the project is: PV_0 = 405.0 $ million. When we incorporate the same options as in the previous examples this we end up in t = 0, with an option value of 54.5 $ million or 11,2% above de base case.

6 Conclusions

In this paper we proposed a Symmetrical Lattice for Real Options valuation that is more generic than the Cox, et al. (1979) model since it is suitable to a wider number of Markov types of stochastic processes. Binomial lattices of this type are not only precise, robust and flexible, but are intuitive enough for Real Options teaching and practitioner alike. The approach proposed has an advantage in modeling over the Cox, et al. lattice in that the variance of the process (2^{nd} moment) is directly modeled in the symmetrical lattice therefore leaving the drift of the process (1^{st} moment) to be modeled independently. This construction scheme may appear more intuitive for the real option learner besides allowing a more generic parameterization of the Markov process that may appear more suited for the application at hand.

We then apply the Symmetrical Lattice model to an expansion real option coupled to an abandonment option, and model the projects cash flows as Geometric Brownian Motion, Geometric Mean Reversion and Geometric Mean Reversion with Drift where the models long term mean is subject to a constant increase rate. The GBM example is compared to the same project modeled through the Cox, et al. approach and we show that both models yield similar results. The other examples show how flexible this approach is and how it can be adapted to a number of distinct types of Markov processes.

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