Evaluating Real Sequential R&D Investment Opportunities

Roger Adkins*

Bradford University School of Management

Dean Paxson**

Manchester Business School

Submitted to Real Options Conference, London

January, 2012

JEL Classifications: D81, G31, H25

Keywords: Real Option Analysis, Multi-stage Multi-factor Sequential Investment, Perpetual Compound Option, Catastrophic Risk

Acknowledgements:

*Bradford University School of Management, Emm Lane, Bradford BD9 4JL, UK.

r.adkins@bradford.ac.uk

+44 (0)1274233466.

**Manchester Business School, Booth St West, Manchester, M15 6PB, UK.

dean.paxson@mbs.ac.uk

+44(0)1612756353.

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Abstract

We provide an analytical solution for American perpetual compound options, that do not rely on a bivariate or multivariate distribution function. This model is especially applicable for a real sequential R&D investment opportunity, such as a series of drug development, tests and clinical trials, where the project can be cancelled at any time, and where the probability of failure declines over stages of completion. The effect of changing input parameter values can clearly be seen in terms of resulting overall project process volatility, and the mark-up factor which justifies continuing with each investment stage. The results are not always intuitive. Some increases in project process failure or in project volatility result in a decrease in the mark-up factor which justifies investments in subsequent stages.

1 Introduction

We extend the result of McDonald and Siegel (1986) for a single investment opportunity to the case of a multiple sequential investment opportunity, while retaining the simplicity of a closed-form solution. Our solution depends on assuming a probability of catastrophic failure at each investment stage that declines in value as the project nears completion, which is a characterisitic of many R&D projects

We conceive a real sequential investment opportunity as a set of distinct, ordered investments that have to be made before the project can be completed. The project can then be interpreted as a collection of investment stages, such that no stage investment, except the first, can be started until the preceding stage has been completed. Success at each stage is not guaranteed because of the possibility of a catastrophic failure that reduces the project value to zero, but if all the stages are successfully completed, then the project value is realized. The following three-stage opportunity provides an illustration: (i) undertaking research and development to create a marketable product, (ii) testing its viability and (iii) implementing the infrastructure for launch and delivery. Bearing in mind that a project can be composed of any number of distinct stages, the form of the illustrated process is common amongst industries as diverse as oil exploration and mining, aircraft manufacture, pharmaceuticals and consumer electronics.

Schwartz and Moon (2000) illustrate a new drug development process which consists of four distinct phases, each with a positive probability of failure, although not necessarily declining over time. Cortazar, Schwartz and Casassus (2003) describe four natural resource exploration stages of a project with technical success probability increasing over each phase, and then a production phase which is subject to commodity price uncertainty. Penning and Sereno (2011) describe a typical development path of a new medicine over seven phases, with a probability of failure declining over time.

Making an investment at a stage depends on whether the prevailing project value is of sufficient magnitude to economically justify committing the investment cost, or whether it is more desirable to wait for more favorable conditions. In our formulation, these conditions are represented by the prevailing project value and investment cost, which are both treated as stochastic, and possibly correlated. After making the stage investment, there is no absolute guarantee that the stage will be successfully completed, because of the presence of irresolvable difficulties in converting intentions into reality owing to technological, technical or market impediments. This means that the stage investment opportunity is subject to a catastrophic failure that causes its value to be entirely destroyed, and the project as an entity is irredeemably lost. Our aim is to analyze this sequential investment opportunity under the three sources of uncertainty, the stochastic project value and the investment cost, and the probability of a catastrophic failure, so to be able to produce a closed-form rule on the investment decision at each of the project stages.

Although the single-stage investment opportunity model of McDonald and Siegel (1986) yields a closed-form solution, this degree of analytical elegance has not be achieved for the multi-stage sequential investment opportunity. Dixit and Pindyck (1994) identify the rule for a two-stage sequential investment but for fixed investment costs. Their solution, based on American perpetuity options, is identical to the one-stage model but with accumulated costs. Nevertheless, it is important to solve the sequential investment problem because amongst other things, the project value may vary between succeeding stages and the option value at each stage needs to be evaluated. Their resolution is an appeal to the time-to-build model of Majd and Pindyck (1987). In this representation, firms can invest continuously, at a rate no greater than a specified maximum, until the project has been completed, but investment may be temporarily halted at any time and subsequently re-started, albeit at a zero cost. The solution, evaluated by using numerical methods, shows the importance of the project value volatility in deciding whether or not to suspend investment activities. Even though the investment levels can be managed, it is essentially a single stage representation. Schwartz and Moon (2000) extend the model by including the possibility of a catastrophic failure and the presence of multiple stages, but their solution again rests on numerical methods.

Other authors simplify the multiple investment stage problem for obtaining a meaningful solution. By assuming a fixed time between stages, Bar-Ilan and Strange (1998) formulate a two-stage sequential investment model and obtain a solution by treating the option as European. Building on the valuation of sequential exchange opportunities by Carr (1988), Lee and Paxson (2001) use an element of European style compound options (and approximation of an American option phase) for formulating a two-stage sequential investment. Brach and Paxson (2001) examine a two-stage sequential investment opportunity similar to the formulation currently under study but they confine their attention more to valuation. Childs and Triantis (1999) formulate a multiple sequential investment model with interaction and obtain a solution through using a trinomial lattice. For all of these expositions, the solution is either not analytical or is restricted to only two stages.

The aim of this paper is to revisit the sequential investment model originally specified by Dixit and Pindyck (1994). Combinations of three distinct sources of uncertainty associated with project value, investment cost and catastrophic failure are proposed as possible contenders for reaching a meaningful solution. Amongst these, we find that the uncertainty, at each stage, regarding the possibility of a catastrophic failure that causes the "sudden death" for the project is absolutely crucial. Although the uncertain project value is normally an essential ingredient of the real option model, it alone cannot yield a meaningful solution as established by Dixit and Pindyck (1994). However, a meaningful solution does arise when the sequential investment opportunity is considered in conjunction with the failure probability. The presence of an uncertain investment cost is not critical to obtaining a meaningful solution, but its inclusion does create a richer representation.

The major analytical findings for the sequential investment model are developed in Section 2. Based on the three sources of uncertainty, the model is presented first for a one-stage opportunity, and then incrementally developed for a two-, three- and finally N-stage sequential investment opportunity. It is assumed that the failure probabilities at successive stages have to decline as the stage nears completion. We develop closed-form solutions for whether or not to commit investment at a particular stage and for the option value at each stage. In Section 3, we obtain further insights into the model's behavior through a numerical illustration. The last section summarizes some advantages and limitations of our model and suggests plausible extensions.

2 Sequential Investment Model

A firm, which can be treated as being a monopolist in its market, is considering an investment project made up of a discrete number of sequential stages, each with a separate investment opportunity. The project as an entity is not fully implemented and the project value realized until all of the sequential stages have been successfully completed. Each successive investment opportunity relies on the successful completion of the preceding investment opportunity. We order each investment opportunity by the number J of stages, including the current one, that are remaining until project completion. Although it may be more natural to label the initial stage of the project as 1, a reverse ordering is used since a backwardation process is used in deriving the solution. First, we examine the decision making position for the ultimate stage where $J = 1$, and then for the preceding stages, incrementally. At the ultimate stage, the firm is considering the decision whether to make an investment in a real asset that renders a future cash flow stream, which rests on the balance between the value of the investment opportunity at $J = 1$ and the net value of the rendered stream. The penultimate stage is labeled $J = 2$. Here, the firm is considering whether to make an expenditure to obtain the investment opportunity at $J = 1$. This decision rests on the balance between the value of the opportunity at $J = 2$ and the value of the opportunity at $J = 1$ less the expenditure made in anticipation of obtaining the option. This procedure is replicated for the stages greater than 2. If the completion of any stage J occurs at time T_J, then $T_J > T_{J+1}$ for all positive integers J since the stages have to be completed consecutively. A representation of the sequential investments process for a $J = N$ stage project is presented in Figure 1. This figure reveals the ordered sequence of stage investments comprising the project. It also shows that after an investment, the possible outcomes are success and failure. If all the stage outcomes are successful, then the project is successfully completed and its value can be realized. However, there is a possibility of failure at each stage. Although the investment is committed, the stage is not successfully completed owing to fundamental irresolvable technical or market impediments, in which case, the project value instantly falls to zero and the project is abandoned without any value. The probability of failure at stage J is denoted by λ_J where $0 \le \lambda_J < 1 \, \forall J$. Situations do arise when an investment expenditure yields an innovative breakthrough and generates an unanticipated increase in the project value, but we have ignored this possibility.

---- Figure 1 about here ----

The value of the project is defined by V. The investment expenditure made at stage J is denoted by K_J for all possible values of J. Both the project value and the set of investment expenditures are treated as stochastic. It is assumed that they are individually well described by the geometric Brownian motion process:

$$\mathrm{d}X = \alpha_X X \mathrm{d}t + \sigma_X X \mathrm{d}z_X \,, \tag{1}$$

for $X \in \{V, K_J \; \forall J\}$, where α_X represent the respective drift parameters, σ_X the respective instantaneous volatility parameter, and $\mathrm{d}z_X$ the respective increment of a standard Wiener process. Dependence between any two of the factors is represented by the covariance term; so, for example, the covariance between the real asset value and the investment expenditure at stage J is specified by:

$$\mathrm{Cov}\left[\mathrm{d}V, \mathrm{d}K_J\right] = \rho_{VK_J} \sigma_V \sigma_{K_J} \mathrm{d}t \,.$$

Different stages may have different factor volatilities and correlations. The riskless rate is r, and the investment expenditure at each stage K is assumed to be instantaneous.

2.1 One-Stage Model

The stage $J = 1$ model represents the investment opportunity for developing a project value V following the investment expenditure K_1, given that the research effort may fail totally with probability λ_1. Although obtainable by directly appealing to McDonald and Siegel (1986), the solution method for this two-factor model follows Adkins and Paxson (2011) because of its extendability to models having more than two factors. The value F_1 of the investment opportunity for $J=1$ depends on the project value and the investment cost, so $F_1 = F_1\left(V, K_1\right)$. By Ito's lemma, the risk neutral valuation relationship is:

$$\begin{aligned}&\tfrac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 F_1}{\partial V^2} + \tfrac{1}{2}\sigma_1^2 K^2 \frac{\partial^2 F_1}{\partial K_1^2} + \rho_{VK_1}\sigma_V\sigma_{K_1} V K_1 \frac{\partial^2 F_1}{\partial V \partial K_1} \\ &\qquad + \theta_V V \frac{\partial F_1}{\partial V} + \theta_{K_1} K_1 \frac{\partial F_1}{\partial K_1} - \left(r + \lambda_1\right) F_1 = 0,\end{aligned} \tag{2}$$

where the θ_X for $X \in \left\{V, K_J \,\forall J\right\}$ denote the respective risk neutral drift rate parameters. The generic solution to (2) is:

$$F_1 = A_1 V^{\beta_1} K_1^{\eta_{11}}, \tag{3}$$

where β_1 and η_{11} denote the generic unknown parameters for the two factors, project value and investment cost, and A_1 denotes a generic unknown coefficient. In this notation, the first subscript for A_1, β_1 and η_{11} refers to the specific stage under consideration, while the second subscript of η_{11} refers to any feasible successive stage. This only becomes relevant for $J > 1$. The valuation function (3) is the solution to (2) with characteristic root equation:

$$\begin{aligned}&Q_1\left(\beta_1, \eta_{11}\right) \\ &= \tfrac{1}{2}\sigma_V^2 \beta_1\left(\beta_1 - 1\right) + \tfrac{1}{2}\sigma_{K_1}^2 \eta_{11}\left(\eta_{11} - 1\right) + \rho_{VK_1}\sigma_V\sigma_{K_1}\beta_1\eta_{11} + \theta_V\beta_1 + \theta_{K_1}\eta_{11} - \left(r + \lambda_1\right) = 0.\end{aligned} \tag{4}$$

The function Q_1 specifies an ellipse defined over a two-dimensional space spanned by two unknown parameters, β_1 and η_{11}. Since for a zero value of one parameter, the other parameter takes on a positive and a negative value, Q_1 has a presence in all 4 quadrants, which we label I – IV. For these four quadrants:

I: $\{\beta_{11}, \eta_{111}\}$ $\beta_{11} \geq 0, \eta_{111} \geq 0$

II: $\{\beta_{12}, \eta_{112}\}$ $\beta_{12} \geq 0, \eta_{112} \leq 0$

III: $\{\beta_{13}, \eta_{113}\}$ $\beta_{13} \leq 0, \eta_{113} \leq 0$

IV: $\{\beta_{14}, \eta_{114}\}$ $\beta_{14} \leq 0, \eta_{114} \geq 0$

This suggests that (3) takes the expanded form:

$$F_1 = A_{11} V^{\beta_{11}} K_1^{\eta_{111}} + A_{12} V^{\beta_{12}} K_1^{\eta_{112}} + A_{13} V^{\beta_{13}} K_1^{\eta_{113}} + A_{14} V^{\beta_{14}} K_1^{\eta_{114}} \tag{5}$$

Now (5) is simplified by invoking the limiting boundary conditions. An economic incentive exists to exercise the investment option at stage $J = 1$ provided that the project value is sufficiently high and the investment cost is sufficiently low, in which case, the option value will be positive. This suggests that the relevant quadrant is II, $\beta_{12} \geq 0,\ \eta_{112} \leq 0$ and $A_{12} > 0$. In contrast, there is no economic incentive to exercise the option if the project value is significantly low or the investment cost is significantly high, in which case, the investment option value will be zero. This implies that $A_{11} = A_{13} = A_{14} = 0$. Then, (5) becomes:

$$F_1 = A_{12} V^{\beta_{12}} K_1^{\eta_{112}}. \tag{6}$$

The threshold levels for the project value and the investment cost signalling the optimal exercise for the investment option at stage $J = 1$ are denoted by $\hat{V}_1$ and $\hat{K}_{11}$, respectively. Then, according to the value matching relationship that the optimal exercise occurs at the balance between the option value $\hat{F}_1 = F_1\left(\hat{V}_1, \hat{K}_1\right)$ and the net value $\hat{V}_1 - \hat{K}_1$, we have:

$$A_{12} \hat{V}^{\beta_{12}} \hat{K}_{11}^{\eta_{112}} = \hat{V}_1 - \hat{K}_{11}. \tag{7}$$

The first order condition for optimality is characterized by the two associated smooth pasting conditions, one for each factor, Samuelson (1965) and Dixit (1993). These can be expressed as:

$$A_{12}\hat{V}_1^{\beta_{12}}\hat{K}_{11}^{\eta_{112}} = \frac{\hat{V}_1}{\beta_{12}} = -\frac{\hat{K}_{11}}{\eta_{112}}. \tag{8}$$

Since the option value is always non-negative, $A_{12} \geq 0$, so $\beta_{12} \geq 0$ and $\eta_{112} < 0$. Together, (7) and (8) demonstrate Euler's result on homogeneity degree-one functions, Sydsæter and Hammond (2006), so $\beta_{12} + \eta_{112} = 1$. Replacing η_{112} by $1 - \beta_{12}$ in (4) yields:

$$Q_1\left(\beta_{12}, 1-\beta_{12}\right) = \tfrac{1}{2}\sigma_1^2\beta_{12}\left(\beta_{12}-1\right) + \beta_{12}\left(\theta_V - \theta_1\right) - \left(r + \lambda_1 - \theta_1\right) = 0, \tag{9}$$

where $\sigma_1^2 = \sigma_V^2 + \sigma_{K_1}^2 - 2\rho_{V,K_1}\sigma_V\sigma_{K_1}$. From (9), β_{12} is the positive root solution for a quadratic equation, which is greater than 1. Further, the threshold levels are related by:

$$\hat{V}_1 = \frac{\beta_{12}}{\beta_{12}-1}\hat{K}_{11}, \tag{10}$$

with $A_{12} = \beta_{12}^{-\beta_{12}}\left(\beta_{12}-1\right)^{\beta_{12}-1}$. Also, the option threshold value at the $J=1$ stage defined by $\hat{F}_1 = F_1\left(\hat{V}_1, \hat{K}_{11}\right)$ is:

$$\hat{F}_1 = \frac{\hat{V}_1}{\beta_{12}}. \tag{11}$$

Applying Ito's lemma to (6), then:

$$\mathrm{d}F_1 = \theta_{F_1}F_1\mathrm{d}t + \sigma_{F_1}F_1\mathrm{d}z_{F_1}, \tag{12}$$

where:

$$\theta_{F_1} = \tfrac{1}{2}\beta_{12}\left(\beta_{12}-1\right)\left\{\sigma_V^2 + \sigma_{K_1}^2 - 2\rho_{VK_1}\sigma_V\sigma_{K_1}\right\} + \beta_{12}\theta_V + \left(1-\beta_{12}\right)\theta_{K_1},$$

$$\sigma_{F_1}^2 = \beta_{12}^2\sigma_V^2 + \left(1-\beta_{12}\right)^2\sigma_{K_1}^2 + 2\beta_{12}\left(1-\beta_{12}\right)\rho_{VK_1}\sigma_V\sigma_{K_1}.$$

Under risk neutrality, the expected return on the option equals the risk-free rate adjusted by the probability of failure, so $\theta_{F_1} = r + \lambda_1$, which is borne out by Q_1, (4).

2.2 Two-Stage Model

At stage $J=2$, the firm examines the viability of making an investment expenditure K_2 to acquire the option to invest F_1 by comparing the value of the compound option F_2 with the net benefits F_1-K_2. Because of (6), F_2 depends on the three factors V, K_1 and K_2, so $F_2=F_2\left(V,K_1,K_2\right)$. By Ito's lemma, the risk neutral valuation relationship for F_2 is:

$$\begin{aligned}&\tfrac{1}{2}\sigma_V^2V^2\frac{\partial^2F_2}{\partial V^2}+\tfrac{1}{2}\sigma_{K_1}^2K_1^2\frac{\partial^2F_2}{\partial K_1^2}+\tfrac{1}{2}\sigma_{K_2}^2K_2^2\frac{\partial^2F_2}{\partial K_2^2}\\&+\rho_{V,K_1}\sigma_V\sigma_{K_1}VK_1\frac{\partial^2F_2}{\partial V\partial K_1}+\rho_{V,K_2}\sigma_V\sigma_{K_2}VK_2\frac{\partial^2F_2}{\partial V\partial K_2}+\rho_{K_1,K_2}\sigma_{K_1}\sigma_{K_2}K_1K_2\frac{\partial^2F_2}{\partial K_1\partial K_2}\\&\qquad+\theta_VV\frac{\partial F_2}{\partial V}+\theta_{K_2}K_2\frac{\partial F_2}{\partial K_2}+\theta_{K_1}K_1\frac{\partial F_2}{\partial K_1}-\left(r+\lambda_2\right)F_2=0.\end{aligned}\tag{13}$$

We conjecture the solution to (13) as a product power function, with generic form:

$$F_2=A_2V^{\beta_2}K_1^{\eta_{21}}K_2^{\eta_{22}},\tag{14}$$

where β_2, η_{21} and η_{22} denote the generic unknown parameters for the three factors, project value and investment expenditure at stage one and two respectively, and A_2 denotes an unknown coefficient. Substitution reveals that (14) is indeed the solution to (13), with characteristic root equation:

$$\begin{aligned}&Q_2\left(\beta_2,\eta_{21},\eta_{22}\right)\\&\quad=\tfrac{1}{2}\sigma_V^2\beta_2\left(\beta_2-1\right)+\tfrac{1}{2}\sigma_{K_1}^2\eta_{21}\left(\eta_{21}-1\right)+\tfrac{1}{2}\sigma_{K_2}^2\eta_{22}\left(\eta_{22}-1\right)\\&\qquad+\rho_{VK_1}\sigma_V\sigma_{K_1}\beta_2\eta_{21}+\rho_{VK_2}\sigma_V\sigma_{K_2}\beta_2\eta_{22}+\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\eta_{21}\eta_{22}\\&\qquad\qquad+\theta_V\beta_2+\theta_{K_1}\eta_{21}+\theta_{K_2}\eta_{22}-\left(r+\lambda_2\right)=0.\end{aligned}\tag{15}$$

The function Q_2 specifies a hyper-ellipse defined over a three dimensional space spanned by the three unknown parameters, β_2, η_{21} and η_{22}. Since any one parameter has both a positive and a negative root for zero values of the remaining two parameters, the hyper-ellipse has a presence in all 8 quadrants. Labeling these quadrants as I – VIII, where:

I	$\{\beta_{21}, \eta_{211}, \eta_{221}\}$	$\beta_{21} \geq 0, \eta_{211} \geq 0, \eta_{221} \geq 0$
II	$\{\beta_{22}, \eta_{212}, \eta_{222}\}$	$\beta_{22} \geq 0, \eta_{212} \geq 0, \eta_{222} < 0$
III	$\{\beta_{23}, \eta_{213}, \eta_{223}\}$	$\beta_{23} \geq 0, \eta_{213} < 0, \eta_{223} \geq 0$
IV	$\{\beta_{24}, \eta_{214}, \eta_{224}\}$	$\beta_{24} \geq 0, \eta_{214} < 0, \eta_{224} < 0$
V	$\{\beta_{25}, \eta_{215}, \eta_{225}\}$	$\beta_{25} < 0, \eta_{215} \geq 0, \eta_{225} \geq 0$
VI	$\{\beta_{26}, \eta_{216}, \eta_{226}\}$	$\beta_{26} < 0, \eta_{216} \geq 0, \eta_{226} < 0$
VII	$\{\beta_{27}, \eta_{217}, \eta_{227}\}$	$\beta_{27} < 0, \eta_{217} < 0, \eta_{227} \geq 0$
VIII	$\{\beta_{28}, \eta_{218}, \eta_{228}\}$	$\beta_{28} < 0, \eta_{218} < 0, \eta_{228} < 0$

The expanded version of the valuation function (14) then becomes:

$$F_2 = \sum_{M=1}^{8} A_{2M} V^{\beta_{2M}} K_1^{\eta_{21M}} K_2^{\eta_{22M}}. \tag{16}$$

The form of (16) is simplified by invoking the limiting boundary conditions. Applying a similar argument as before reveals the relevant quadrant to be IV. Exercising the option F_2 is economically justified only if the project value V is sufficiently high and the investment expenditures, K_1 and K_2, are sufficiently low, while the resulting option value F_2 only becomes significantly high provided that $\beta_2 \geq 0$, $\eta_{21} < 0$ and $\eta_{22} < 0$. In contrast, there is no economic justification for exercising the option F_2 whenever the project value is sufficiently low, or either of the two investment expenditures, K_1 and K_2, are sufficiently high. This suggests that the quadrants other than IV are not relevant, and that their coefficients, A_{21}, A_{22}, A_{23}, A_{25}, A_{26}, A_{27} and A_{28}, are all set to equal zero. Consequently, (16) simplifies to:

$$F_2 = A_{24} V^{\beta_{24}} K_1^{\eta_{214}} K_2^{\eta_{224}}. \tag{17}$$

The option at stage $J=2$ is exercised when the option value F_2 is balanced by the net value for the acquired option $F_1 - K_2$. If the thresholds at exercise for the project value and the two investment expenditures for the $J=1$ and the $J=2$ stages are denoted by $\hat{V}_2$, $\hat{K}_{12}$ and $\hat{K}_{22}$, respectively, then the value matching relationship becomes:

$$A_{24}\hat{V}_2^{\beta_{24}}\hat{K}_{12}^{\eta_{214}}\hat{K}_{22}^{\eta_{224}} = A_{12}\hat{V}_2^{\beta_{12}}\hat{K}_{12}^{1-\beta_{12}} - \hat{K}_{22}, \tag{18}$$

where A_{12} and β_{12} are known from the stage $J=1$ evaluation. The three smooth pasting conditions associated with (18), one for each of the three factors, can be expressed as:

$$\beta_{24}A_{24}\hat{V}_2^{\beta_{24}}\hat{K}_{12}^{\eta_{214}}\hat{K}_{22}^{\eta_{224}} = \beta_{12}A_{12}\hat{V}_2^{\beta_{12}}\hat{K}_{12}^{1-\beta_{12}}, \tag{19}$$

$$\eta_{214}A_{24}\hat{V}_2^{\beta_{24}}\hat{K}_{12}^{\eta_{214}}\hat{K}_{22}^{\eta_{224}} = \left(1-\beta_{12}\right)A_{12}\hat{V}_2^{\beta_{12}}\hat{K}_{12}^{1-\beta_{12}}, \tag{20}$$

$$\eta_{224}A_{24}\hat{V}_2^{\beta_{24}}\hat{K}_{12}^{\eta_{214}}\hat{K}_{22}^{\eta_{224}} = -\hat{K}_{22}. \tag{21}$$

Since an option value is non-negative, then $A_{24} \geq 0$. This implies that $\beta_{24} \geq 0$ from (19), $\eta_{214} < 0$ from (20), and $\eta_{224} < 0$ from (21), which corroborate our finding on the signs of the parameters. Moreover, the dependence amongst the parameters can be found from combining the smooth pasting conditions and the value matching relationship. First, a comparison of (19) and (21) with (18) yields:

$$\eta_{224} = 1 - \frac{\beta_{24}}{\beta_{12}}, \tag{22}$$

which implies that $\beta_{24} > \beta_{12}$. Second, a comparison of (19) with (20) yields:

$$\eta_{214} = \frac{1-\beta_{12}}{\beta_{12}}\beta_{24}. \tag{23}$$

Third, a comparison of (22) with (23) yields:

$$\beta_{24} + \eta_{214} + \eta_{224} = 1. \tag{24}$$

The equation (24) implies that F_2 is a homogeneity degree-one function, so it follows that the valuation function (14) can be expressed as:

$$F_2 = B_{24} \left[F_1 \left(V, K_1 \right) \right]^{\phi_{24}} K_2^{1-\phi_{24}}, \tag{25}$$

where $B_{24} = A_{24} A_{12}^{-\phi_{24}}$ and $\phi_{24} = \beta_{24} / \beta_{12} > 1$. This implies that the value matching relationship (18) can be expressed as a two-factor model where the value of the option to invest at stage $J = 2$ is represented by a homogeneous degree-one function. For this two-factor model, we denote the thresholds for the investment expenditure by $\hat{K}_{22}$ and for the stage $J = 1$ option value by $\hat{F}_{12} = F_{12} \left(\hat{V}_2, \hat{K}_{12} \right)$. Then from (18) we have:

$$B_{24} \hat{F}_{12}^{\phi_{24}} \hat{K}_{22}^{1-\phi_{24}} = \hat{F}_{12} - \hat{K}_{22}, \tag{26}$$

where $B_{24} = \phi_{24}^{-\phi_{24}} \left(\phi_{24} - 1 \right)^{\phi_{24}-1}$. Except for change in variables, (26) has the identical form as (7), so it follows that:

$$\hat{F}_{12} = \frac{\phi_{24}}{\phi_{24} - 1} \hat{K}_{22} = \frac{\beta_{24}}{\beta_{24} - \beta_{12}} \hat{K}_{22}. \tag{27}$$

This states that an investment made at the $J = 2$ stage is only economically justified provided that the value of the option to invest at the $J = 1$ stage exceeds the investment expenditure at the $J = 2$ stage. This finding extends the standard result of McDonald and Siegel (1986) for the single stage investment opportunity model to the case of a two-stage sequential investment opportunity model. Further, (27) implies that the boundary discriminating between making an investment at the $J = 2$ stage and not making an investment is characterized by a linear proportional line relating $\hat{F}_{12}$ and $\hat{K}_{22}$.

Determining the investment expenditure threshold $\hat{K}_{22}$ at the $J=2$ stage requires knowledge of the option threshold $\hat{F}_{12}$ to invest at the $J=1$ stage. Now, owing to (11), $\hat{F}_{12}=\hat{V}_2/\beta_{12}$, so (27) becomes:

$$\hat{V}_2=\frac{\beta_{12}\beta_{24}}{\beta_{24}-\beta_{12}}\hat{K}_{22}. \tag{28}$$

This asserts that for an economically justified investment to be made at the $J=2$ stage, the project value at that stage has to exceed the investment cost at that stage. This finding again represents an extension of the result for the single stage investment opportunity model. Also, the boundary signaling investment at the $J=2$ stage is specified by a linear function linking $\hat{V}_2$ and $\hat{K}_{22}$.

If the threshold option value at the $J=2$ stage is denoted by $\hat{F}_2=F_2\left(\hat{V}_2,\hat{K}_{12},\hat{K}_{22}\right)$, then from (19) we have:

$$\hat{F}_2=\frac{\beta_{12}\hat{F}_{12}}{\beta_{24}}=\frac{\hat{V}_2}{\beta_{24}}. \tag{29}$$

The option value and discriminatory boundary are evaluated from knowing the values of the parameters β_{24} and ϕ_{24}. After eliminating η_{214} and η_{224} by using (23) and (22), respectively, then (15) becomes:

$$\begin{aligned}&Q_2\left(\beta_{12}\phi_{24},\left(1-\beta_{12}\right)\phi_{24},1-\phi_{24}\right)\\&\quad=\tfrac{1}{2}\phi_{24}\left(\phi_{24}-1\right)\sigma_2^2+\phi_{24}\left\{\tfrac{1}{2}\beta_{12}\left(\beta_{12}-1\right)\sigma_1^2+\beta_{12}\left(\theta_V-\theta_{K_1}\right)+\left(\theta_{K_1}-\theta_{K_2}\right)\right\}\\&\qquad-\left(r+\lambda_2-\theta_{K_2}\right)=0,\end{aligned} \tag{30}$$

where β_{12} has been evaluated at the $J=1$ stage, and:

$$\sigma_2^2 = \beta_{12}^2\sigma_V^2 + \left(1-\beta_{12}\right)^2\sigma_{K_1}^2 + \sigma_{K_2}^2$$
$$+2\beta_{12}\left(1-\beta_{12}\right)\rho_{VK_1}\sigma_V\sigma_{K_1} - 2\beta_{12}\rho_{VK_2}\sigma_V\sigma_{K_2} - 2\left(1-\beta_{12}\right)\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}.$$

Standard real option theory tell us that the underlying volatility has a profound effect on the solution. Accordingly, a positive change in σ_2 produces a decrease in the parameter ϕ_{24} but an increase in the mark-up factor $\phi_{24}/\left(\phi_{24}-1\right)$. Now, σ_2 depends on the parameter β_{12} as well as the volatilities for V, K_1 and K_2, and their covariances. We first consider the consequences if all the covariances can be assumed to be zero. High values for β_{12}, which are caused by low σ_V and σ_{K_1}, tend to ratchet up the value of σ_2, while a value of β_{12} closer to 1 due to high σ_V or σ_{K_1}, tends to diminish the effect of σ_{K_1} in explaining σ_2. The importance of σ_{K_1} in determining σ_2 depends on its magnitude relative to σ_V. Further, since the value of β_{12} depends positively on the probability of a catastrophic failure at the $J=1$ stage, the importance of σ_{K_2} relative to σ_V and σ_{K_1} in explaining σ_2 diminishes as the failure probability increases. It is through this mechanism that the probabilities of catastrophic failures at succeeding stages are translated into the investment strategy at the current stage.

We now turn our attention to the effects of the covariance terms on σ_2. If $\rho_{VK_1} > 0$, $\rho_{VK_2} > 0$, or $\rho_{K_1K_2} < 0$, then the value of σ_2 declines while the value of β_{12} increases relative to the instance of zero correlations. This can be explained in the following way. A long investment cost acts as a partial hedge for a long project value whenever $\rho_{VK_1} > 0$ and $\rho_{VK_2} > 0$, since a random positive (negative) movement in the investment cost is partly compensated by a movement in the same direction in the project value. (A long/short position in the investment cost might be established through fixed-price/cost-plus construction contracts). This partial hedge reduces the riskiness of the combined position, which is reflected in a lower value of σ_2. In contrast, a long investment cost and V position becomes more risky whenever ρ_{VK_1} or ρ_{VK_2} is negative. If $\rho_{K_1K_2} < 0$, then a random movement in the $J=2$ stage investment cost tends to be followed by a movement in the opposite direction in the $J=1$ stage investment cost, and a long $J=2$ stage investment cost acts as a hedge against a short $J=1$ stage investment cost. A positive movement in the $J=2$ stage investment cost that is followed by a negative movement in the $J=1$ stage investment cost can be interpreted as dynamic learning, since a higher than anticipated preliminary investment cost leads to a lower investment cost at a subsequent stage, while a negative movement in the $J=2$ stage investment cost that is followed by a positive movement in the $J=1$ stage investment cost can be interpreted as compensatory. Under-investment is corrected by over-investment at a subsequent stage. In contrast, when $\rho_{K_1K_2} > 0$, the volatility σ_2 is inflated. This can arise from a positive movement in the $J=2$ stage investment cost that is followed by a positive movement in the $J=1$ stage investment cost, which suggests that errors at the earlier $J=2$ stage are compounded at the later $J=1$ stage. However, a positive value for $\rho_{K_1K_2}$ can just as well be due to a negative movement in the $J=2$ stage investment cost followed by a negative movement in the $J=1$ stage investment cost. This may also represent bad news if low investment levels presage low project values. Clearly, the sensitivity of the volatility σ_2 depends on the magnitudes of the contributory quantities as well as their interactions.

Using (4) to replace β_{12}, (30) becomes:

$$\begin{aligned} &Q_2\left(\beta_{12}\phi_{24},(1-\beta_{12})\phi_{24},1-\phi_{24}\right) \\ &\quad = \tfrac{1}{2}\phi_{24}\left(\phi_{24}-1\right)\sigma_2^2+\phi_{24}\left(r+\lambda_1-\theta_{K_2}\right)-\left(r+\lambda_2-\theta_{K_2}\right)=0. \end{aligned} \tag{31}$$

The parameter ϕ_{24} is evaluated as the positive root of (31), which is required to be greater than one. From (31), we know Q_2 is a quadratic function of ϕ_{24}. Given that $\sigma_2^2 > 0$, since it is a variance expression, then $\phi_{24} > 1$ provided that the value of Q_2 evaluated at $\phi_{24} = 1$ is negative, Dixit and Pindyck (1994). It can be observed from (31) that for $Q_2 < 0$, then $\lambda_2 > \lambda_1$, see Figure 2.

---- *Figure 2 about here* ---

The parameter λ measures the conditional probability of a catastrophic failure at a particular stage. The existence of a solution to the sequential investment model represented by an American perpetual compound option depends crucially on the probabilities at the two stages following a distinct pattern. Although it plays an important role in deciding an acceptable investment level at each stage, the stochastic nature of the investment expenditures is not critical, since a model solution exists even when σ_{K_1} and σ_{K_2} are both set equal to zero provided σ_2 remains positive. For a solution to our two-stage sequential investment model to exist, the only requirement is that the conditional probability of a failure at the $J=2$ stage has to exceed that for the $J=1$ stage. This condition can be seen simply as a stipulation imposed by the model structure. Since $\lambda_2 > (1-\lambda_2)\lambda_1$, the failure probability at the $J=2$ stage is always greater than that for the $J=1$ stage. Alternatively, this condition can be interpreted as the presence of dynamic learning. Because of the reduction in the failure probabilities, the effect of making an investment at the $J=2$ stage is to increase the affordable amount of investment expenditure at the subsequent stage. Ceteris paribus, project viability is able to support a higher level of investment expenditure at the next stage, and this implies some element of learning.

The condition $\lambda_2 > \lambda_1$ for obtaining an economically meaningful solution does not require the investment cost at each stage to be necessarily stochastic. The model continues to yield a sensible result even if σ_{K_1} and σ_{K_2} are zero so our findings apply for a deterministic investment cost. It follows that for a meaningful solution to emerge, the probabilities of a catastrophic failure at each stage have to follow a specific pattern, and not that the investment levels have to be stochastic.

By applying Ito's lemma to (25), then the return on the option F_2 is:

$$\frac{\mathrm{d}F_2}{F_2} = \theta_{F_2}\mathrm{d}t + \sigma_{F_2}\mathrm{d}z_{F_2}, \tag{32}$$

where:

$$\theta_{F_2} = \tfrac{1}{2}\phi_{24}\left(\phi_{24}-1\right)\left\{\sigma_{F_1}^2 + \sigma_{K_2}^2 - 2\rho_{F_1K_2}\sigma_{F_1}\sigma_{K_2}\right\} + \phi_{24}\theta_{F_1} + \left(1-\phi_{24}\right)\theta_{K_2} ,$$

$$\rho_{F_1K_2}\sigma_{F_1}\sigma_{K_2} = \beta_{12}\rho_{VK_2}\sigma_V\sigma_{K_2} + \left(1-\beta_{12}\right)\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2} ,$$

$$\sigma_{F_2}^2 = \phi_{24}^2\sigma_{F_1}^2 + \left(1-\phi_{24}\right)^2\sigma_{K_2}^2 + 2\phi_{24}\left(1-\phi_{24}\right)\rho_{F_1K_2}\sigma_{F_1}\sigma_{K_2} .$$

Under risk neutrality, $\theta_{F_2} = r + \lambda_2$, which is borne out by the Q_2 function.

2.3 Three-Stage Model

Since the extension of the sequential investment model to the $J = 3$ stage is achieved by replication, we only provide the crucial results with only a basic explanation. Then, the comparison of the results for each of the three stages facilitates the formulation of the general result for the $J = N$ stage.

The value of the option to invest at the $J = 3$ stage F_3 depends on the project value V, and the investment costs at the $J = 1$, $J = 2$ and $J = 3$ stages, K_1, K_2 and K_3, respectively, so $F_3 = F_3\left(V, K_1, K_2, K_3\right)$. Using Ito's lemma, it can be shown that the risk neutral valuation relationship for F_3 is a four-dimensional partial differential equation, whose solution is the product power function:

$$F_3 = A_3 V^{\beta_3} K_1^{\eta_{13}} K_2^{\eta_{23}} K_3^{\eta_{33}} , \tag{33}$$

with characteristic root equation:

$$
\begin{aligned}
&Q_3\left(\beta_3,\eta_{13},\eta_{23},\eta_{33}\right)\\
&=\tfrac{1}{2}\sigma_V^2\beta_3\left(\beta_3-1\right)+\tfrac{1}{2}\sigma_{K_1}^2\eta_{13}\left(\eta_{13}-1\right)+\tfrac{1}{2}\sigma_{K_2}^2\eta_{23}\left(\eta_{23}-1\right)+\tfrac{1}{2}\sigma_{K_3}^2\eta_{33}\left(\eta_{33}-1\right)\\
&\quad+\rho_{VK_1}\sigma_V\sigma_{K_1}\beta_3\eta_{13}+\rho_{VK_2}\sigma_V\sigma_{K_2}\beta_3\eta_{23}+\rho_{VK_3}\sigma_V\sigma_{K_3}\beta_3\eta_{33}\\
&\quad+\rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\eta_{13}\eta_{23}+\rho_{K_1K_3}\sigma_{K_1}\sigma_{K_3}\eta_{13}\eta_{33}+\rho_{K_2K_3}\sigma_{K_2}\sigma_{K_3}\eta_{23}\eta_{33}\\
&\quad+\theta_V\beta_3+\theta_{K_1}\eta_{13}+\theta_{K_2}\eta_{23}+\theta_{K_3}\eta_{33}-\left(r+\lambda_3\right)=0.
\end{aligned}
\tag{34}
$$

The function Q_3 specifies a hyper-ellipse that has a presence in all possible quadrants. The relevant quadrant is that where $\beta_3 > 0$, $\eta_{13} < 0$, $\eta_{23} < 0$ and $\eta_{33} < 0$. For convenience, we suppress the subscript designating the relevant quadrant.

The value matching relationship reflects value conservation at the $J = 3$ stage. The threshold levels signaling an investment at the $J = 3$ stage for V, K_1, K_2 and K_3 are denoted by $\hat{V}_3$, $\hat{K}_{13}$, $\hat{K}_{23}$ and $\hat{K}_{33}$, respectively. Since the value matching relationship balances the option value to invest at the $J = 3$ stage to the option value to invest at the $J = 2$ stage net of the investment cost, all evaluated at the $J = 3$ stage threshold levels, then:

$$\hat{F}_3 = \hat{F}_{23} - \hat{K}_{33} \tag{35}$$

where $\hat{F}_3 = A_3 \hat{V}_3^{\beta_3} \hat{K}_{13}^{\eta_{13}} \hat{K}_{23}^{\eta_{23}} \hat{K}_{33}^{\eta_{33}}$ and $\hat{F}_{23} = F_2\left(\hat{V}_3, \hat{K}_{13}, \hat{K}_{23}\right) = A_2 \hat{V}_3^{\beta_2} \hat{K}_{13}^{\eta_{12}} \hat{K}_{23}^{\eta_{22}} - \hat{K}_{33}$. Because of the homogeneity degree-one property, (35) can be expressed as:

$$B_3 \hat{F}_{23}^{\phi_3} K_{33}^{1-\phi_3} = \hat{F}_{23} - \hat{K}_{33}\,. \tag{36}$$

By comparing the power parameters for $\hat{F}_3$ and $\hat{F}_{23}^{\phi_3} K_{33}^{1-\phi_3}$, then it can be established that:

$$
\begin{aligned}
&\beta_3 = \phi_3\beta_2 = \phi_3\phi_2\phi_1,\\
&\eta_{13} = \phi_3\eta_{12} = \phi_3\phi_2\left(1-\phi_1\right),\\
&\eta_{23} = \phi_3\eta_{22} = \phi_3\left(1-\phi_2\right),\\
&\eta_{33} = \left(1-\phi_3\right).
\end{aligned}
$$

where $\phi_1 = \beta_1$. It follows from (36) that the optimal condition signaling an investment at the $J=3$ stage is:

$$\hat{F}_{23} = \frac{\phi_3}{\phi_3 - 1}\hat{K}_{33}. \tag{37}$$

Now, from (29):

$$\hat{F}_{23} = \frac{\hat{V}_3}{\beta_2} = \frac{\hat{V}_3}{\phi_2\phi_1},$$

so (37) can be rewritten as:

$$\hat{V}_3 = \frac{\phi_3\phi_2\phi_1}{\phi_3 - 1}\hat{K}_{33}. \tag{38}$$

From (35), the smooth pasting condition with respect to the project value V can be expressed as $\beta_3\hat{F}_3 = \beta_2\hat{F}_{23}$, so:

$$\hat{F}_3 = \frac{\hat{V}_3}{\phi_3\phi_2\phi_1}. \tag{39}$$

In (38), the solution to the boundary discriminating between investing and not investing at the $J=3$ stage requires evaluating ϕ_3, since ϕ_2 and ϕ_1 are each calculated at the subsequent stages. By eliminating η_{13}, η_{23} and η_{33} from (34) yields after some simplification:

$$\begin{aligned}
&Q_3\left(\phi_3\phi_2\phi_1, \phi_3\phi_2\left(1-\phi_1\right), \phi_3\left(1-\phi_2\right), \left(1-\phi_3\right)\right) \\
&\quad = \tfrac{1}{2}\sigma_3^2\phi_3\left(\phi_3 - 1\right) \\
&\qquad + \phi_3\left[\tfrac{1}{2}\sigma_2^2\phi_2\left(\phi_2 - 1\right) + \phi_2\left\{\tfrac{1}{2}\sigma_1^2\phi_1\left(\phi_1 - 1\right) + \theta_V\phi_1 + \theta_{K_1}\left(1-\phi_1\right)\right\} + \theta_{K_2}\left(1-\phi_2\right) - \theta_{K_3}\right] \\
&\qquad - \left(r + \lambda_3 - \theta_{K_3}\right) = 0,
\end{aligned}$$

where:

$$
\begin{aligned}
\tfrac{1}{2}\sigma_3^2 = {} & \tfrac{1}{2}\sigma_V^2\phi_2^2\phi_1^2 + \tfrac{1}{2}\sigma_{K_1}^2\phi_2^2\left(1-\phi_1\right)^2 + \tfrac{1}{2}\sigma_{K_2}^2\left(1-\phi_2\right)^2 + \tfrac{1}{2}\sigma_{K_3}^2 \\
& + \rho_{VK_1}\sigma_V\sigma_{K_1}\phi_1\left(1-\phi_1\right)\phi_2^2 + \rho_{VK_2}\sigma_V\sigma_{K_2}\phi_1\phi_2\left(1-\phi_2\right)\beta_3\eta_{23} - \rho_{VK_3}\sigma_V\sigma_{K_3}\phi_1\phi_2 \\
& + \rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\left(1-\phi_1\right)\phi_2\left(1-\phi_2\right) - \rho_{K_1K_3}\sigma_{K_1}\sigma_{K_3}\left(1-\phi_1\right)\phi_2 - \rho_{K_2K_3}\sigma_{K_2}\sigma_{K_3}\left(1-\phi_2\right).
\end{aligned}
$$

Then after further simplification, we obtain:

$$
Q_3 = \tfrac{1}{2}\sigma_3^2\phi_3\left(\phi_3-1\right) + \phi_3\left(r+\lambda_2-\theta_{K_3}\right) - \left(r+\lambda_3-\theta_{K_3}\right) = 0. \tag{40}
$$

The value of ϕ_3 is the positive root solution to (40), which exceeds one provided that $\lambda_3 > \lambda_2$.

2.4 N-Stage Model

The solution to the $J = N > 1$ stage of the sequential investment model is derived by extrapolating the results for the $J = 2$ and $J = 3$ stages. The value of the option to invest at the $J = N$ stage, denoted by $F_N = F_N\left(V, K_1, \ldots, K_N\right)$, is described by a $N+1$ dimensional partial differential equation, whose solution takes the form of a product power function:

$$
F_N = A_N V^{\beta_N} K_1^{\eta_{1N}} K_2^{\eta_{2N}} \ldots K_N^{\eta_{NN}}. \tag{41}
$$

The values for the power parameters are given by:

$$
\begin{aligned}
\beta_N &= \phi_N\phi_{N-1}\ldots\phi_3\phi_2\phi_1, \\
\eta_{1N} &= \phi_N\phi_{N-1}\ldots\phi_3\phi_2\left(1-\phi_1\right), \\
\eta_{2N} &= \phi_N\phi_{N-1}\ldots\phi_3\left(1-\phi_2\right), \\
&\vdots \\
\eta_{N-1N} &= \phi_N\left(1-\phi_{N-1}\right), \\
\eta_{NN} &= \left(1-\phi_N\right).
\end{aligned} \tag{42}
$$

The relationship between the threshold levels for V and K_N, denoted by $\hat{V}_N$ and $\hat{K}_{NN}$, respectively, is given by:

$$
\hat{V}_N = \frac{\phi_N\ldots\phi_3\phi_2\phi_1}{\phi_N - 1}\hat{K}_{NN}. \tag{43}
$$

The value of ϕ_N for $N > 1$ is evaluated as the positive root of the equation:

$$Q_N\left(\phi_N\right) = \tfrac{1}{2}\sigma_N^2\phi_N\left(\phi_N - 1\right) + \phi_N\left(r + \lambda_{N-1} - \theta_{K_N}\right) - \left(r + \lambda_N - \theta_{K_N}\right) = 0, \tag{44}$$

where the full expression for σ_N^2 is specified by:

$$\begin{aligned}
\tfrac{1}{2}\sigma_N^2 = {} & \tfrac{1}{2}\sigma_V^2\phi_{N-1}^2\phi_{N-2}^2\ldots\phi_2^2\phi_1^2 + \tfrac{1}{2}\sigma_{K_1}^2\phi_{N-1}^2\phi_{N-2}^2\ldots\phi_2^2\phi_2^2\left(1-\phi_1\right)^2 + \tfrac{1}{2}\sigma_{K_2}^2\phi_{N-1}^2\phi_{N-2}^2\ldots\phi_3^2\left(1-\phi_2\right)^2 \\
& + \ldots + \tfrac{1}{2}\sigma_{K_{N-2}}^2\phi_{N-1}^2\left(1-\phi_{N-2}\right)^2 + \tfrac{1}{2}\sigma_{K_{N-1}}^2\left(1-\phi_{N-1}\right)^2 + \tfrac{1}{2}\sigma_{K_N}^2 \\
& + \rho_{VK_1}\sigma_V\sigma_{K_1}\phi_{N-1}^2\ldots\phi_3^2\phi_2^2\phi_1\left(1-\phi_1\right) + \rho_{VK_2}\sigma_V\sigma_{K_2}\phi_{N-1}^2\ldots\phi_3^2\phi_2\left(1-\phi_2\right)\phi_1 \\
& + \ldots + \rho_{VK_{N-1}}\sigma_V\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\phi_{N-2}\ldots\phi_2\phi_1 \\
& \quad - \rho_{VK_N}\sigma_V\sigma_{K_N}\phi_{N-1}\phi_{N-2}\ldots\phi_2\phi_1 \\
& + \rho_{K_1K_2}\sigma_{K_1}\sigma_{K_2}\phi_{N-1}^2\ldots\phi_3^2\phi_2\left(1-\phi_2\right)\left(1-\phi_1\right) \\
& \quad + \rho_{K_1K_3}\sigma_{K_1}\sigma_{K_3}\phi_{N-1}^2\ldots\phi_4^2\phi_3\left(1-\phi_3\right)\phi_2\left(1-\phi_1\right) \\
& + \ldots + \rho_{K_1K_{N-2}}\sigma_{K_1}\sigma_{K_{N-2}}\phi_{N-1}^2\phi_{N-2}\left(1-\phi_{N-2}\right)\phi_{N-3}\ldots\phi_2\left(1-\phi_1\right) \\
& \quad + \rho_{K_1K_{N-1}}\sigma_{K_1}\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\phi_{N-2}\ldots\phi_2\left(1-\phi_1\right) \\
& \quad - \rho_{K_1K_N}\sigma_{K_1}\sigma_{K_N}\phi_{N-1}\phi_{N-2}\ldots\phi_2\left(1-\phi_1\right) \\
& + \rho_{K_2K_3}\sigma_{K_2}\sigma_{K_3}\phi_{N-1}^2\ldots\phi_4^2\phi_3\left(1-\phi_3\right)\left(1-\phi_2\right) \\
& \quad + \rho_{K_2K_4}\sigma_{K_2}\sigma_{K_4}\phi_{N-1}^2\ldots\phi_5^2\phi_4\left(1-\phi_4\right)\phi_3\left(1-\phi_2\right) \\
& + \ldots + \rho_{K_2K_{N-2}}\sigma_{K_2}\sigma_{K_{N-2}}\phi_{N-1}^2\phi_{N-2}\left(1-\phi_{N-2}\right)\phi_{N-3}\ldots\phi_3\left(1-\phi_2\right) \\
& \quad + \rho_{K_2K_{N-1}}\sigma_{K_2}\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\phi_{N-2}\ldots\phi_3\left(1-\phi_2\right) \\
& \quad - \rho_{K_2K_N}\sigma_{K_2}\sigma_{K_N}\phi_{N-1}\phi_{N-2}\ldots\phi_3\left(1-\phi_2\right) \\
& + \ldots \\
& + \rho_{K_{N-3}K_{N-2}}\sigma_{K_{N-3}}\sigma_{K_{N-2}}\phi_{N-1}^2\left(1-\phi_{N-2}\right)\left(1-\phi_{N-3}\right) \\
& \quad + \rho_{K_{N-3}K_{N-1}}\sigma_{K_{N-3}}\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\phi_{N-2}\left(1-\phi_{N-3}\right) \\
& \quad - \rho_{K_{N-3}K_N}\sigma_{K_{N-3}}\sigma_{K_N}\phi_{N-1}\phi_{N-2}\left(1-\phi_{N-3}\right) \\
& + \rho_{K_{N-2}K_{N-1}}\sigma_{K_{N-2}}\sigma_{K_{N-1}}\phi_{N-1}\left(1-\phi_{N-1}\right)\left(1-\phi_{N-2}\right) \\
& \quad - \rho_{K_{N-2}K_N}\sigma_{K_{N-2}}\sigma_{K_N}\phi_{N-1}\left(1-\phi_{N-2}\right) \\
& - \rho_{K_{N-1}K_N}\sigma_{K_{N-1}}\sigma_{K_N}\left(1-\phi_{N-1}\right),
\end{aligned}$$

The volatility for each stage can be expressed more succinctly by:

$$\sigma_N^2 = \mathrm{w}^{\mathrm{T}}\Omega\mathrm{w} \tag{45}$$

where Ω is the $N+1$ dimensional square variance-covariance matrix with its first diagonal element being σ_V^2, the second $\sigma_{K_1}^2$, and so on until $\sigma_{K_N}^2$. The off-diagonal elements denote the corresponding covariances. The column vector w is given by:

$$\mathrm{w} = \begin{bmatrix} \phi_{N-1} \cdots \phi_3 \phi_2 \phi_1 \\ \phi_{N-1} \cdots \phi_3 \phi_2 \left(1-\phi_1\right) \\ \phi_{N-1} \cdots \phi_3 \left(1-\phi_2\right) \\ \vdots \\ \phi_{N-1}\left(1-\phi_{N-2}\right) \\ \left(1-\phi_{N-1}\right) \\ -1 \end{bmatrix}$$

Because of the homogeneity degree-one property, we have $\mathrm{w}^{\mathrm{T}}\mathbf{i}=0$ where $\mathbf{i}$ is the unit vector. For $N=1$, $\mathrm{w}^{\mathrm{T}}=\left[1,-1\right]$.

3 Numerical Illustrations

We have presented a general analytical framework for evaluating a sequential investment project that can be characterized by $J=N$ successive investment stages. This framework incorporates three sources of uncertainty: (i) uncertainty regarding the asset value on completion of the project, (ii) uncertainty regarding the cost of the investment at each of the stages, and (iii) uncertainty, at each stage, regarding the possibility of a catastrophic failure that causes the "sudden death" of the project. Further, the framework allows for the first two types of uncertainty to co-vary. Analysis of the general framework also yields closed-form solutions for the optimal investment strategy as the maximal amount of investment allowable at each stage according to the project's prevailing value. We establish that for an investment to be economically justified at each stage, the prevailing project value has to exceed the anticipated investment cost. Moreover that the probability of a catastrophic failure at each stage has to decline successively as the stage approaches completion.

To obtain additional insights into the behaviour of the analytical framework, we conduct some numerical evaluations on an illustration involving a 4-stage sequential investment project using the base case information exhibited in Table 1. The set of probabilities of catastrophic failure at the stages adheres to the condition $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. Initially, the variances for the investment costs at the four stages have been set to be equal and the covariance terms between the five factors to equal zero. These are altered for the sensitivity analysis.

---- Table 1 about here ----

First, we consider the results for the base case information, and then examine the impact of key sensitivities.

Table 2 shows the results calculated from the values exhibited in Table 1, using the backwardation principle so the $J = 1$ stage is enumerated first, then the $J = 2$ stage, and so on. The volatilities at each of the 4 stages, σ_1, σ_2, σ_3 and σ_4, are evaluated from (45), the parameters ϕ_J for $J = 1$ from (9) and for $J = 2, 3, 4$ from (44), and the mark-up factors for each of the 4 stages from (43). It can be seen from Table 2 that the volatilities at each stage increase in value as the stage in question becomes more distant from completion. This finding is in line with expectations, since the volatility depends not only on the volatilities for the project value and the current stage investment cost but also on the cascading effect of the investment cost volatilities and parameter values for all possible subsequent stages. As expected, the parameter values ϕ_J for $J = 1, 2, 3, 4$ are all greater than one. This feature arises owing to the pattern of failure probabilities specified in Table 1. Even though $\lambda_4 > \lambda_3 > \lambda_2 > \lambda_1$, as required by the model, this does not imply that the ϕ_J necessarily follow a declining pattern. The values for the ϕ_J depend not only on the volatility for the stage in question but also on the failure probability λ_J. These two effects work in opposing directions. While an increase in volatility for the J stage yields an increase in ϕ_J, an increase in the failure probability λ_J leads a decline in ϕ_J. Although the former is more dominant according Dixit and Pindyck (1994), it is conceivable that $\phi_{J+1} > \phi_J$, as Table 2 illustrates.

---- *Table 2 about here* ----

The last column in Table 2 presents the mark-up factor. This reveals the mark-up factors to be in excess of one, as required. According to our results, the mark-up factors increase in magnitude as the stage becomes more distant from completion, but there is no theoretical requirement for this. The mark-up factor is a critical element of the investment strategy, since it stipulates for each stage that for an investment at that stage to be economically justified, the ratio of the anticipated project value to investment cost has to be at least equal to the mark-up factor. If we can assume equal project values at each stage, which is unlikely because of their differences in timing, then the overall mark-up factor is given by:

$$\left[\frac{1}{\phi_1}+\frac{1}{\phi_2}+\frac{1}{\phi_3}+\frac{1}{\phi_4}\right]^{-1} = 1.3007 .$$

For the project to remain economically viable, the maximum total investment cost cannot exceed 76.9% of the project value.

3.1 Probability of Failure

We examine how the probability of a catastrophic failure influences the solution initially by increasing its value, λ_J for $J = 1, 2, 3, 4$ by a constant amount of 5%. The results, which are presented in Table 3, conform with expectations. The increase in probability at each stage has the consequence of raising the volatility, of lowering the parameter value ϕ_J and raising the mark-up factor, but only for stages 2, 3 and 4. This contrasts with the result for the $J = 1$ stage. For the first stage, since an increase in λ_1 effectively raises the discount rate but leaves the volatility σ_1 unaffected, there is a consequential rise in the parameter value ϕ_J and fall in the mark-up factor reflecting a greater urgency in exercising the option, Dixit and Pindyck (1994). However, this feature is not replicated for stages $J = 2,3,4$ since the effect of the increased discount rate due to the failure probability increase is dominated by the cascading impact of the failure probability at the stage $J = 1$ on the volatility σ_J at stages $J = 2,3,4$. A change in the failure probability λ_J for $J = I$ has both a direct effect on the solution at the stage $J = I$ as well as an indirect effect on the solution at the stages $J > I$ due to the cascading impact of λ_I on the volatilities $\sigma_{J>I}$.

---- *Table 3 about here* ----

The comparative magnitudes of the direct and indirect effects can be ascertained by increasing the failure probability only at a single stage. This is illustrated in Table 4, which is divided into three separate panels. Panel A displays the solution for a failure probability increase of 5% only at the stage $J = 1$, Panel B only at stage $J = 2$, and Panel C only at stage $J = 3$. The direct effect, for each panel of results, conforms to the already observed pattern of a rise in the parameter value and a fall in the mark-up factor, contemporaneous with the failure probability increase. Further, the rise in the parameter value seems to vary according to the proportional rather than absolute change in the failure probability. More interestingly, the indirect effect only endures for the immediate preceding stage and rapidly evaporates for the stages previous to that. So, a failure probability increase at the $J = 1$ stage impacts significantly on the $J = 2$ stage solution, but leaves the $J = 3$ and $J = 4$ stage solution almost unaffected, see Table 2 and Panel A of Table 4. This pattern for the indirect effect for the $J = 1$ stage is replicated for a failure probability increase both at the $J = 2$ and $J = 3$ stages, see Panels B and C of Table 4. Since the indirect effect is characterized by a parameter value decrease but a mark-up factor increase, an anticipated failure probability increase occurring at the next stage J acts as an investment deterrent at the current stage $J + 1$.

---- *Table 4 about here* ----

3.2 Volatility

For the single-stage investment opportunity, an increase in project value volatility is normally accompanied with a fall in the parameter value and a rise in the mark-up factor. We obtain this finding for the multi-stage sequential investment opportunity but only for the final investment, when $J=1$. This can be observed from Table 5, which illustrates the effect of increasing the project value volatility to 40% on the solution. For the remaining stages, $J=2,3,4$, while there is a fall in the parameter value, there is, in contrast, also a fall in the mark-up factor due to the attenuating effect of the subsequent parameter values on the mark-up factor, see equation (43). A similar pattern of effects arising from the increase in project value volatility is replicated for an increase in the investment cost volatility. Table 6 illustrates the impact of increasing the investment cost volatility to 10% at the stage $J=1$ on the solution. It can be seen that this change leads to a fall in the parametric value and a rise in the mark-up factor at the stage $J=1$, but a rise in the parameter value and a fall in the mark-up factor at the stages $J=2,3,4$.

---- Tables 5 and 6 about here ----

3.3 Correlation

Changes in the correlation coefficients impact on the solution through the relevant stage volatility, σ_J for $J=1,2,3,4$, which in turn influences the parameter value. Further, since the volatility at the preceding stage σ_{J+1} depends on the volatility at the current stage σ_J, changes in the correlation coefficient cascade through the volatilities of the preceding stages. Theoretically, we argue that owing to the hedging effect, a positive change in the correlation between the project value and the investment cost depresses the stage volatility, which in turn raises the parameter value, while a negative change in the correlation between two separate stage investment costs depresses the stage volatility. The primary aim of the sensitivity analysis is to corroborate these finding. Although we only consider the cases of a positive project value and investment cost correlation and a negative correlation between investment costs, the results we obtain are nevertheless representative.

Table 7 illustrates the effects on the solution when the correlation between the project value and the stage $J = 1$ investment cost is increased to 50%. As expected, there is a fall in the stage $J = 1$ volatility, which produces an increase in the parameter value ϕ_1 but a decrease in the mark-up factor. Since the investment cost acts as a form of hedge for the project value, a greater economically justified level in the investment cost is able to be sustained by the project value. However, the decline in the stage $J = 1$ does not cascade into the volatilities at the preceding stages, σ_J for $J = 2,3,4$. In contrast, Table 7 shows that there is an increase in the volatilities at the stages $J = 2,3,4$, which is explained by the role of the parameter ϕ_1 in determining the volatility at these stages. For the stages $J = 2,3,4$, the volatility is observed to rise, while there is a fall in the parameter value but a rise in the mark-up factor. It seems that the improvement in the stage $J = 1$ mark-up factor is being compensated by increases in the mark-up factor at the stages $J = 2,3,4$. The pattern of results that an increase in the correlation leads to a decrease in the contemporaneous stage volatility, but an increase in the preceding stage volatilities is replicated when, for example, the correlation between the project value and the stage $J = 2$ investment cost is increased while ρ_{VK_1} is set to its base case value of zero.

---- Table 7 about here ----

The cascade effect observed in Table 7 is not sustained for positive increases in all of the correlations between the project value and the investment cost at each of the four stages. By setting $\rho_{VK_1} = \rho_{VK_2} = \rho_{VK_3} = \rho_{VK_4} = 50\%$, Table 8 illustrates the effects of a correlation increase on the solution. This reveals that the correlation increase is accompanied by a fall in the volatility at all of the stages, $J = 1,2,3,4$, which leads to a rise in the parameter value and a fall in the mark-up factor. For this kind of correlation increase, the decrease in the stage volatility arising from the positive correlation dominates the cascade effect observed in Table 7. If a positive hedge is present at each of the sequential stages, then this is reflected in lower values for the mark-up factor for each stage.

---- Table 8 about here ----

When we consider a negative change in the correlation between the investment costs, the findings tend to conform to the pattern observed in Tables 7 and 8. Table 9 illustrates the effect on the solution of decreasing the correlation $\rho_{K_1K_2}$ from its base case value to -50%, while all the other input parameter values remain unchanged. This reveals that the stage $J=1$ solution is unaffected by this shift, because of the absence of any dependence between σ_1 and $\rho_{K_1K_2}$, while at the stages $J=2,3,4$, there is a change in solution because of the dependence between the respective stage volatilities and $\rho_{K_1K_2}$. Table 9 shows that the effect of setting $\rho_{K_1K_2}=-50\%$ on the stage $J=2$ solution is to lower the stage volatility, to raise the parameter value and to lower the mark-up factor. In contrast, the correlation change produces a positive impact on the volatilities at stages $J=3,4$, which leads to a fall in the respective parameter value and a rise in the mark-up factor. The impact of a change in investment cost correlation at one stage has cascaded into the solution at preceding stages. Now, a negative investment cost correlation indicates the presence of some compensating mechanism, since an unanticipated fall (rise) in investment cost at one stage is associated with a rise (fall) at the other stage. This compensating mechanism seems to have been, in part, projected onto the solution and the mark-up factors for the various stages. If we now consider a positive change in the correlation between investment costs, then we obtain results similar in form but different in sign.

---- *Table 9 about here* ----

The cascade effect observed in Table 9 is not sustained if the investment cost correlations for all possible pairs of stages are set to be equal but different from zero. Table 10 illustrates the effects on the solution of changing the investment cost correlations, $\rho_{K_IK_J}$ for $I,J=1,2,3,4$ with $I \neq J$, to -50%. This reveals that the stage $J=1$ solution remains unchanged, as before. For all the remaining stages, $J=2,3,4$, there is a fall in the stage volatility, a rise in the parameter value and a fall in the mark-up factor. A negative investment cost correlation for all possible stages implies a greater economically justified level for the investment cost for an unchanged project value. If the investment cost correlations for all possible stages are set to be positive, then we obtain a similar form of solution except that the quantities have the opposite sign.

---- Table 10 about here ----

4 Conclusion

We provide an analytical solution for a multi-factor, multi-phase sequential investment process, where there is the real option at any stage of continuing, or abandoning the project development. This model is particularly appropriate for real sequential R&D investment opportunities, such as geological exploration in natural resources that may be followed by development and then production, or drug development processes, where after drug discovery there are subsequent tests and trials required before production and marketing is feasible or allowed. Also, in these cases often there is a decreasing probability of project failure, as more information appears, and the efficacy and robustness of the original discovery are examined.

Other authors have provided unsatisfactory solutions to similar problems, or relied on bivariate or multivariate distribution functions, or required complex numerical solutions.

An advantage of our approach is that the effect of changing input parameter values can clearly be seen in terms of resulting overall project process volatility, and the mark-up factor which justifies continuing with the investment stages. The results are not always intuitive. For instance, an increase in the failure probability by a constant amount for all stages first increases and then eventually decreases the mark-up factor. Increasing the failure probability stage by stage, even with failure decreasing with stage completion, sometimes results in mark-factor reductions. Increasing project value volatility does not always result in a rise of the mark-up factor (a hurdle for continuing the investment process); indeed the mark-up factor sometimes falls in subsequent stages after the first stage investment, as project volatility increases. An increase in the project value and investment cost correlation results in a decrease in the mark-up factor for the first stage, but a significant increase in the mark-up factor for subsequent stages.

So in general, the effect of changes in input parameter values on the real option value at exercise and on the investment process continuance is often surprising, and dependent on the specific input values and the number and sequence of stages, seen only in the solutions for each case.

Our model is not appropriate where the probability of failure increases with completion of each investment stage. Also we have assumed instantaneous investment completion, constant project value and investment cost drifts, volatilities and correlation, and no competition. Relaxing these assumptions are challenging issues for further research.

Figure 1

Sequential Investment Process

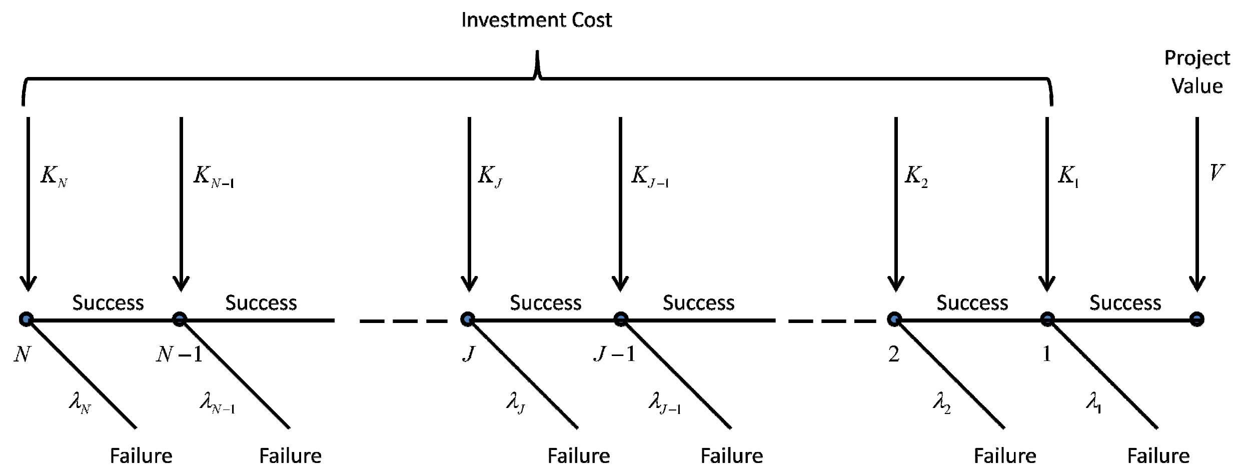

Figure 2

The Q_2 Function for Investment Stage 2

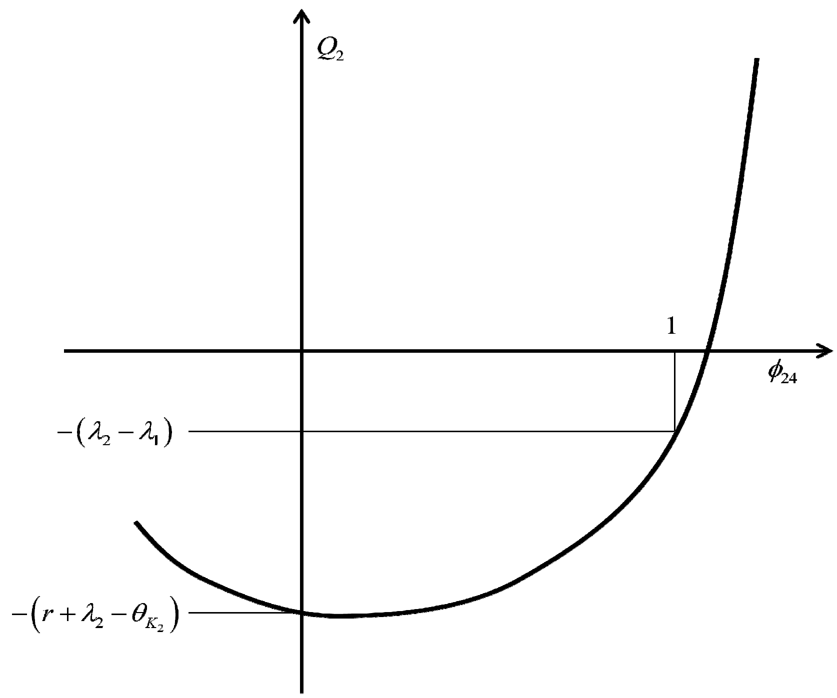

Table 1

Base Case Information

Project value drift rate	θ_V	0%
Project value volatility	σ_V	25%
Investment cost drift rate	$\theta_{K_1} = \theta_{K_2} = \theta_{K_3} = \theta_{K_4}$	0%
Investment cost volatility	$\sigma_{K_1} = \sigma_{K_2} = \sigma_{K_3} = \sigma_{K_4}$	5%
Stage 1 failure probability	λ_1	0%
Stage 1 failure probability	λ_2	10%
Stage 1 failure probability	λ_3	20%
Stage 1 failure probability	λ_4	40%
Risk-free probability	r	6%

All the correlations between the project value and the investment costs at each stage are set to equal zero, so the correlation matrix is specified by:

	V	K_1	K_2	K_3	K_4
V	100%				
K_1	0%	100%			
K_2	0%	0%	100%		
K_3	0%	0%	0%	100%	
K_4	0%	0%	0%	0%	100%

Table 2

Base Case Results

Stage	Volatility	Parameter Value	Mark-up Factor
1	$\sigma_1 = 0.2550$	$\phi_1 = 1.9478$	2.0551
2	$\sigma_2 = 0.4918$	$\phi_2 = 1.4294$	6.4836
3	$\sigma_3 = 0.7015$	$\phi_3 = 1.2176$	15.5797
4	$\sigma_4 = 0.8535$	$\phi_4 = 1.2760$	15.6744

Table 3

The Effect of Increasing the Failure Probability by a Constant Amount

Stage	Probability	Volatility	Parameter Value	Mark-up Factor
1	5%	25.50%	2.4065	1.7110
2	15%	60.78%	1.2875	10.7761
3	25%	78.16%	1.1757	20.7320
4	45%	91.85%	1.2401	18.8159

Table 4

The Effect of Increasing the Failure Probability Stage by Stage

Panel A:

Stage	Probability	Volatility	Parameter Value	Mark-up Factor
1	5%	25.50%	2.4065	1.7110
2	10%	60.78%	1.1547	17.9648
3	20%	70.12%	1.2177	15.5434
4	40%	85.32%	1.2761	15.6404

Panel B:

Stage	Probability	Volatility	Parameter Value	Mark-up Factor
1	0%	25.50%	1.9478	2.0551
2	15%	49.18%	1.5936	5.2294
3	20%	78.18%	1.0920	36.8576
4	40%	85.34%	1.2760	15.6706

Panel C:

Stage	Probability	Volatility	Parameter Value	Mark-up Factor
1	5%	25.50%	1.9478	2.0551
2	10%	49.18%	1.4294	6.4836
3	25%	70.15%	1.3109	11.7405
4	40%	91.87%	1.1852	23.3619

Table 5

The Effect of Increasing the Project Value Volatility to 40%

Stage	Volatility	Parameter Value	Mark-up Factor
1	40.31%	1.4942	3.0234
2	60.03%	1.3331	5.9797
3	79.92%	1.1856	12.7219
4	94.71%	1.2445	12.0230

Table 6

The Effect of Increasing the Investment Cost Volatility at Stage 1 to 10%

Stage	Volatility	Parameter Value	Mark-up Factor
1	26.93%	1.8803	2.1360
2	48.08%	1.4413	6.1414
3	69.14%	1.2213	14.9573
4	84.38%	1.2795	15.1496

Table 7

The Effect of Increasing the Correlation between the Project Value

and the Investment Cost at Stage 1 to 50%

Stage	Volatility	Parameter Value	Mark-up Factor
1	22.91%	2.0924	1.9154
2	50.05%	1.4203	7.0707
3	70.94%	1.2147	16.8115
4	86.12%	1.2732	16.8243

Table 8

The Effect of Increasing the Correlation between the Project Value

and the Investment Cost at Stages 1 - 4 to 50%

Stage	Volatility	Parameter Value	Mark-up Factor
1	22.91%	2.0924	1.9154
2	47.37%	1.4492	6.7500
3	68.49%	1.2237	16.5867
4	83.74%	1.2819	16.8743

Table 9

The Effect of Decreasing the Correlation between Investment Costs

at Stages 1 and 2 to -50%

Stage	Volatility	Parameter Value	Mark-up Factor
1	25.50%	1.9478	2.0551
2	48.94%	1.4320	6.4566
3	70.17%	1.2175	15.6123
4	85.37%	1.2759	15.7049

Table 10

The Effect of Decreasing the Correlation between Investment Costs

at Stages 1 - 4 to -50%

Stage	Volatility	Parameter Value	Mark-up Factor
1	25.50%	1.9478	2.0551
2	48.94%	1.4320	6.4566
3	69.97%	1.2182	16.5700
4	85.02%	1.2772	16.6582

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