Valuation of Real Options by the Gradient Projection Method

Changhyun Kwon • Terry Friesz • Yongma Moon • Tao Yao • Taeil Kim

chkwon@psu.edu • *tfriesz@psu.edu* • *yongma@psu.edu* • *taoyao@psu.edu* • *tzk115@psu.edu*

Abstract

Most real options models are American-type options involving a free boundary problem which can be modeled in the form of variational inequalities. In this paper, we provide a viable mathematical formulation and promising computational approach for the valuation of real options. We study an equivalent optimization problem with an inequality constraint and boundary conditions, whose necessary conditions are the variational inequality articulation of an American option. The objective functional is defined in a Hilbert space and includes a partial differential operator. We propose gradient projection methods for both infinite and finite time horizon real option problems, while we approximate the partial derivatives by finite differences. We test the performance and accuracy of the proposed algorithm, and compare with an existing method, the projective successive over-relaxation.

1 Introduction

The real options approach has become a workhorse in modern economics and Finance. However, many real options studies have focused on relative simple option model. While this type of model has been successful in literature, real problems may involve more complex and realistic situations.

A conventional approach to solve real options problem involving a free boundary is to use value matching, smooth pasting and high contract conditions (Karatzas and Shreve [12], Dixit and Pindyck [6], Dumas [7], Guo et al. [11], Yao [23]). This approach reduces a real option problem to tractable nonlinear equations and is particular useful to derive closed form solutions. However, it is difficult to apply this approach to real options problem when more practical situations such as finite time horizon and generalized Ito process are considered (Nagae and Akamatsu [16]).

In this paper, we study an equivalent optimization problem with an inequality constraint and boundary conditions, whose necessary condition for the optimality is the variational inequality presentation of real options. Our approach considers the practical aspects of real world problems and provides an efficient computational algorithm.

The remainder of this paper is structured as follows. This section concludes with a brief review of the literature, while §2 describes the option to invest problem with finite time horizon and its variational inequality formulation. Section 3 provides an equivalent formulation as the form of an extremal problem. Section 4 studies the gradient projection algorithm, and section 5 presents the numerical results. Section 6 and 7 analyze the investment problem for infinite time horizon and provide the numerical results. Section 8 concludes.

The free boundary problem involving in the valuation of an American option occurs in many systems. This property was first pointed out by McKean [14]. The valuation of American options with dividends has been studied by many scholars. Geske [8], Roll [18], and Whaley [20] derived analytical solutions for the case of known discrete dividends. Finite difference approximation approach, which includes log-transformation, was introduced by Brennan and Schwartz [1] and Brennan and Schwartz [2]. This approach approximates differential terms of the value function by discretizing the time and the state space. The finite difference method is one of popular methods because the method is easily implemented and flexible, so that non-standard forms of options can be computed.

Cox et al. [5] introduced the binomial method for the valuation of American options. The binomial method is implemented easily and flexibly by the discretizations of the both spaces as well. Geske and Shastri [10] summarized and compared these early methods. Later, Geske and Johnson [9] presented an analytic solution to American put option with or without dividends. However, their formula is an infinite series that has to be approximated by numerical methods. Kim [13] and Carr et al. [4] provided an integral representation of the option price. These methods are compared by Broadie and Detemple [3], who also derived the lower bound and upper bound for the value of American options.

2 Optimal Timing to Invest within Finite Time Horizon

In this section, we analyze the option to invest, a typical real option problem, and show that the real option problem can be transformed to a linear complementarity problem (LCP). The LCP is then articulated into a variational inequality (VI). A gradient projection method will be developed to solve this VI and to obtain investment strategy.

We develop a simple option to invest model as in Pindyck [17]. Let $S(t)$ be the project cash flow after the investment.

$$\frac{dS}{S} = (r-\delta)dt + \sigma dW$$

for $r > \delta > 0$, where W is a standard Wiener process, r is the rate of interest, δ is the continuous time dividend rate, and σ is the volatility of the asset price.

Denote the investment cost by K, the option value to invest by $C(S)$.

$$C(S,t) = \max_{\tau\in[t,T]} E\left[e^{-r(\tau-t)}(S(\tau)-K)^+\right]$$

where T is the expiration time for the option to invest. In addition, $C(S,t)$ need to satisfy the following boundary conditions:

$$\begin{aligned} C(0,t) &= 0 \\ \lim_{S\to\infty} C(S,t) &= \infty \\ C(S,T) &= (S(T)-K)^+ \end{aligned}$$

Simply, using Ito's lemma, $C(S,t)$ should satisfy:

$$\mathcal{L}_{BS}(C(S,t)) = \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta)S\frac{\partial C}{\partial S} - rC \le 0 \tag{1}$$

2.1 Variational Inequality Formulation

It is well known that an American call option can be stated as a linear complementarity problem (LCP):

$$\mathcal{L}_{BS}(C(S,t)) \cdot [C(S,t) - \Phi(S)] = 0 \tag{2}$$

$$C(S,t) - \Phi(S) \geq 0 \tag{3}$$

$$-\mathcal{L}_{BS}(C(S,t)) \geq 0 \tag{4}$$

with boundary conditions

$$\begin{aligned} C\,(S,t) &\geq \Phi(S) \\ C\,(S,T) &= \Phi(S) \\ C\,(0,t) &= 0 \\ \lim_{S\to\infty} C\,(S,t) &\to \infty \end{aligned}$$

where we defined the Black-Scholes operator as

$$\mathcal{L}_{BS} \equiv \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-\delta)S\frac{\partial}{\partial S} - r$$

and the pay-off function as

$$\Phi(S) = \max(S - E, 0)$$

Now we wish to formulate an American call option in a *variational inequality problem.* We define the class of variational inequality problems as following:

Definition 1 *Given a nonempty set, Ω, and a function, $F : \Omega \to \mathbb{R}^n$, the variational inequality problem $VIP(F,\Omega)$ is to find a vector y such that*

$$\begin{aligned} y &\in \Omega \\ \langle F(y), x - y\rangle &\geq 0 \qquad \forall x \in \Omega \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the corresponding inner product.

Let us define a set of functions

$$\Omega = \{G(S,t) | G(S,t) - \Phi(S) \geq 0 \quad S \in \mathbb{R}_+, t \in [0,T]\},$$

and pick $U \in \Omega$ so that

$$-\mathcal{L}_{BS}(C(S,t)) \cdot [U(S,t) - \Phi(S,t)] \geq 0 \qquad \forall S \in \mathbb{R}_+, t \in [0,T]. \tag{5}$$

We have also from (2)

$$-\mathcal{L}_{BS}(C(S,t)) \cdot [C(S,t) - \Phi(S)] = 0 \tag{6}$$

Subtraction (6) from (5), we get

$$-\mathcal{L}_{BS}(C(S,t)) \cdot [U(S,t) - C(S,t)] \geq 0 \qquad \forall S \in \mathbb{R}_+ \quad \forall t \in [0,T], \tag{7}$$

or, equivalently,

$$\int_0^\infty -\mathcal{L}_{BS}(C(S,t)) \cdot [U(S,t) - C(S,t)]\, dS \geq 0 \qquad \forall t \in [0,T], \quad (8)$$

which is a variational inequality formulation of an American call option.

2.2 Log Transformation

Let us consider the following transformation:

$$\begin{aligned} y &\equiv \log S \\ \tau &\equiv T - t \\ u(y,\tau) &\equiv C(S,t), \end{aligned}$$

Then (1) becomes

$$-\frac{\partial u}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2} + \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y} - ru \leq 0. \quad (9)$$

Defining an operator

$$\Psi = \frac{\partial}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} - \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial y} + r,$$

and the payoff function

$$\phi(y) = \max(e^y - E, 0),$$

,we obtain a linear complementarity problem and variational inequality problem for this case; it is to find $u(y,\tau)$ such that, for all $v \in \Omega$

$$\Psi(u)\,[u - v] = 0, \Psi(u) \geq 0, u - v \geq 0$$

and the variational inequality problem is to find $u \in \Omega$ for each time instant $\tau \in [0,T]$ such that

$$\int_{-\infty}^{\infty} \Psi(u)\,[v - u]\, dy \geq 0 \quad \forall v \in \Omega, \quad (10)$$

where

$$\Omega = \{v : v - \phi \geq 0, v(y,0) = \phi(y), \qquad v(-\infty,\tau) = \phi(-\infty), v(\infty,\tau) = \phi(\infty)\}. \quad (11)$$

Note we denoted $f(\infty) = \lim_{x\to\infty} f(x)$. In the next section, we will develop an equivalent reformulation as of the form in an extremal problem.

3 An Extremal Problem in Continuous Time for Finite Time Horizon Problems

Now we are interested in the articulation and direct solution of a functional mathematical program whose solutions are also solutions of (10). We show through numerical examples that such an approach is numerically efficient. Consider the extremal problem:

$$\min J\left(u\right)=\int_{-\infty}^{\infty}\int_{0}^{u}\Psi\left(v\right)dvdy \quad \text{s.t.} \quad u\in\Omega \tag{12}$$

where Ω is defined as (11). By deriving a necessary condition for this extremal problem, we recover variational inequality (10), thereby verifying that any solution of (12) is also a solution of (10). Therefore, any solution to (12), provided one exists, is a solution to the linear complementarity problem.

We will need some results of functional analysis to derive the necessary condition. First, we introduce the *Gateaux-differentiability.*

Definition 2 *A functional J is* Gateaux differentiable *or* G-differentiable *at $v\in V$ in the direction $\varphi\in V$, if the limit*

$$\lim_{\theta\to 0}\frac{J(v+\theta\varphi)-J(v)}{\theta}$$

exists. This limit is denoted by $\delta J(v,\varphi)$.

The famous Riesz's representation theorem is presented without proof (see, for example, [22]):

Theorem 3 *Let V be a Hilbert space and $L\in V^*$ a continuous linear form on V. Then there exists a unique element $u_L\in V$ such that*

$$\forall v\in V: \qquad L(v)=\langle u_L,v\rangle$$

and

$$\parallel L\parallel_{V^*}=\parallel u_L\parallel_V .$$

Conversely, we can associate with each $u\in V$ the continuous linear form L_u defined by

$$\forall v\in V: \qquad L_u(v)=\langle u,v\rangle .$$

Using Definition 2 and Theorem 3, we provide the following result::

Theorem 4 *Let V be a Hilbert space. If J is G-differentiable at $v\in V$, and if $\delta J(v,\varphi)$ is a continuous linear form with respect to φ, then, there exists an element $\frac{\partial J}{\partial v}\in V$ such that*

$$\forall\varphi\in V: \qquad \delta J(v,\varphi)=\left\langle\frac{\partial J}{\partial v},\varphi\right\rangle .$$

Moreover, $\frac{\partial J}{\partial v}$ is called the gradient of J at v.

Proof. It is immediate from Theorem 3. ■

The validity of the extremal problem (12) may now be established. To this end, we must establish that the functional $J(u)$ is G-differentiable and the set Ω is convex. The relevant results are:

Lemma 5 *The functional*

$$J(u) = \int_{-\infty}^{\infty} \int_{0}^{u} \Psi(v) dv dy$$

is everywhere G-differentiable and

$$\frac{\partial J(u)}{\partial u} = \Psi(u)$$

Proof. We construct the G-derivative as

$$\begin{aligned}
\delta J(u, \varphi) &= \lim_{\theta \to 0} \frac{J(u+\theta\varphi) - J(u)}{\theta} \\
&= \int_{-\infty}^{\infty} \lim_{\theta \to 0} \frac{\int_0^{u+\theta\varphi} \Psi(v) dv - \int_0^u \Psi(v) dv}{\theta} dy \\
&= \int_{-\infty}^{\infty} \lim_{\theta\varphi \to 0} \frac{\int_0^{u+\theta\varphi} \Psi(v) dv - \int_0^u \Psi(v) dv}{\theta\varphi} \varphi dy \\
&= \int_{-\infty}^{\infty} \Psi(u) \varphi dy
\end{aligned}$$

Since

$$\delta J(u, \varphi) \doteq \left\langle \frac{\partial J}{\partial u}, \varphi \right\rangle \doteq \int_{-\infty}^{\infty} \frac{\partial J}{\partial u} \varphi dy,$$

we have

$$\frac{\partial J}{\partial u} = \Psi(u)$$

■

To give the equivalence of the extremal problem and the variational inequality problem we will need:

Lemma 6 *The set Ω defined by (11) is convex.*

Proof. Pick $\bar{v}, \hat{v} \in \Omega$ so that

$$\bar{v} - \phi \geq 0, \qquad \hat{v} - \phi \geq 0$$

and define

$$v^{\lambda} = \lambda \bar{v} + (1-\lambda)\hat{v} \qquad \lambda \in [0, 1].$$

Then $v^{\lambda} \in \Omega$. ■

Finally, we obtain the following theorem:

Theorem 7 *Any solution of the extremal problem (12) is a solution of the variational inequality (10).*

Proof. Let $v \in \Omega$ be arbitrary. Since Ω is convex, and $u \in \Omega$ implies

$$u + \theta(v - u) \in \Omega \qquad \forall \theta \in [0, 1].$$

Hence for u to be a minimum of J on Ω it is necessary that $\forall v \in \Omega$

$$\left[\frac{d}{d\theta} J(u + \theta(v - u))\right]_{\theta=0} = \delta J(u, v - u) \geq 0.$$

Since J is G-differentiable at u and δJ is well-defined by Lemma 5, we have

$$\delta J(u, v - u) = \int_{-\infty}^{\infty} \Psi(u)(v - u)dy \geq 0 \qquad \forall v \in \Omega.$$

(10) follows immediately. ■

By the above theorem, the solution for the extremal problem obtained by any method is indeed a solution to the variational inequality problem.

4 The Gradient Projection Algorithm in Hilbert Spaces

We study in this section the following projected gradient method:

$\boxed{\textit{Step 0. Initialization.}}$ *Set* $k = 0$. *Pick* $u^0(y, \tau) \in \Omega$.

$\boxed{\textit{Step 1. Determine gradient.}}$ *Calculate*

$$\begin{aligned} \frac{\partial J^k}{\partial u} \equiv \frac{\partial J\left(u^k\right)}{\partial u} &= \Psi(u^k) \\ &= \left[\frac{\partial u^k}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 u^k}{\partial y^2} \right. \\ &\quad \left. - \left(r - \delta - \frac{1}{2}\sigma^2\right) \frac{\partial u^k}{\partial y} + r u^k \right] \end{aligned}$$

$\boxed{\textit{Step 3. Update iterate.}}$ *Calculate*

$$u^{k+1} = P_\Omega \left\{ u^k - \theta_k \frac{\partial J^k}{\partial u} \right\} = \max \left\{ \phi, u^k - \theta_k \frac{\partial J^k}{\partial u} \right\}$$

where P_Ω *denotes the minimum norm projection onto* Ω *and* θ_k *is a variable scalar step.*

$\boxed{\textit{Step 4. Stopping test.}}$ *If an appropriate stopping test is satisfied, halt execution and declare*

$$u^*(y, \tau) \approx u^{k+1}(y, \tau)$$

Otherwise set $k = k + 1$ *and go to Step 1.*

For the convergence of this scheme and the detailed discussion, see the Chapter 10 in [15].

4.1 Finite Difference Approximation and Time Stepping

Now we are interested in a finite approximation of infinite dimensional variational inequality problem (10). To recall

$$\int_{-\infty}^{\infty} \Psi(u)\,[v-u]\,dy \geq 0 \quad \forall v \in \Omega, \qquad \forall \tau \in [0,T] \tag{13}$$

We limit the domain of space y by an interval $[y_L, y_U]$ instead of $(-\infty, \infty)$ and discretize the interval by M sub intervals so that

$$\begin{aligned} y_i &= y_L + i\delta y, \qquad i = 0, ..., M \\ \delta y &= \frac{y_U - y_L}{M} \end{aligned}$$

Also we discretize the time by L intervals so that

$$\begin{aligned} \tau_j &= j\delta\tau, \qquad j = 0, ..., L \\ \delta\tau &= \frac{T}{L} \end{aligned}$$

Then (13) is approximated to

$$\sum_{i=0}^{M} \Psi(u_{i,j})\,[v_{i,j} - u_{i,j}] \geq 0 \quad \forall v \in \Omega, \qquad \forall j \in \{0, ..., L\} \tag{14}$$

where $u_{i,j} = u\,(y_i, \tau_j)$. By its nature, the finite difference approximation has an instability property which depends on the mesh sizes, δy and $\delta\tau$. Because of the higher derivatives with respect to y, M should be much bigger than L, which means more meshes on y-axis.

Note that, for $j = 0$, that is $\tau = 0$, the VI (14) has the solution $u_{i,0} = \phi_i = \phi\,(y_i)$ from the initial condition. Starting from this solution for $j = 0$, we may solve (14) for the entire time domain step by step. Our next interest is, of course, how to approximate the parabolic operator $\Psi\,(\cdot)$. We may consider following approximations

$$\begin{aligned} \frac{\partial u_{i,j}}{\partial \tau} &\approx \frac{u_{i,j} - u_{i,j-1}}{\delta\tau} \\ \frac{\partial u_{i,j}}{\partial y} &\approx \theta\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\delta y}\right) \\ &\quad + (1-\theta)\left(\frac{u_{i+1,j-1} - u_{i-1,j-1}}{2\delta y}\right) \\ \frac{\partial^2 u_{i,j}}{\partial y^2} &\approx \theta\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\delta y)^2}\right) \\ &\quad + (1-\theta)\left(\frac{u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}}{(\delta y)^2}\right) \end{aligned}$$

where we used θ-approximation for the derivatives with respect to the space y. For $\theta =$

$0, \frac{1}{2}, 1$, the approximation becomes explicit, Crank-Nicolson, and implicit, respectively.

$$\begin{aligned}
\Psi\left(u_{i,j}; u_{i,j-1}\right) &\approx \frac{u_{i,j} - u_{i,j-1}}{\delta\tau} \\
&- \frac{1}{2}\sigma^2 \left[\theta\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\delta y)^2}\right)\right. \\
&\qquad \left. + (1-\theta)\left(\frac{u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}}{(\delta y)^2}\right)\right] \\
&- \left(r - \delta - \frac{1}{2}\sigma^2\right)\left[\theta\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\delta y}\right)\right. \\
&\qquad \left. + (1-\theta)\left(\frac{u_{i+1,j-1} - u_{i-1,j-1}}{2\delta y}\right)\right] \\
&+ r u_{i,j}
\end{aligned}$$

where we denote $\Psi\left(u_{i,j}; u_{i,j-1}\right)$ the approximation of the operator $\Psi(\cdot)$ at $u_{i,j}$ given $u_{i,j-1}$ for all i and j.

As discussed above, the algorithm will be of the form:

1. For $j = 0$, have the solution of the VI (14), that is, $u_{i,0} = \phi_i = \phi(y_i)$. Set $j = 1$.
2. Given the solution $u_{i,j-1}$ for the current value of j and for all i, solve the following VI

$$\sum_{i=0}^{M} \Psi(u_{i,j}; u_{i,j-1})\left[v_{i,j} - u_{i,j}\right] \geq 0 \quad \forall v_{i,j} \in \Omega \tag{15}$$

   to obtain $u_{i,j}$ for the current j and for all i.
3. Set $j = j + 1$ and repeat Step 2 until $j = L$.

In this paper, the VIP (15) will be solved by the gradient projection method for the equivalent extremal problem.

4.2 Estimation of the Optimal Timing to Invest

In an optimal timinig to invest, i.e., for $S > S^*$, the option value $C(S,t)$ becomes $\Phi(S) = \max(E - S, 0)$. By this argument, we can have an estimate for the optimal timing by computing S^* such that

$$\left|C\left(S^*, t\right) - \Phi\left(S^*\right)\right| < \varepsilon$$

for a given $t \in [0, T]$ and the pre-set tolerance $\varepsilon > 0$.

5 Numerical Results

We have tested the gradient projection algorithm with discretizations both in time and space for several American call options. Broadie and Detemple [3] studied upper and lower bounds for the values of American option, with which we compared our result in Table 1. The binomial method with 15,000 steps ([3]) is used to compare. To have accurate result, in

Option Parameters	Asset Price	Lower Bound	Upper Bound	Binomial Method	Gradient Projection
$r = 0.03$	80.000	0.218	0.220	0.219	0.218
$\sigma = 0.20$	90.000	1.376	1.389	1.386	1.382
$\delta = 0.07$	100.000	4.750	4.792	4.783	4.777
	110.000	11.049	11.125	11.098	11.093
	120.000	20.000	20.061	20.000	20.001
$r = 0.03$	80.000	2.676	2.691	2.689	2.678
$\sigma = 0.40$	90.000	5.694	5.727	5.722	5.708
$\delta = 0.07$	100.000	10.190	10.250	10.239	10.223
	110.000	16.110	16.201	16.181	16.166
	120.000	23.271	23.392	23.360	23.347
$r = 0.00$	80.000	1.029	1.039	1.037	1.033
$\sigma = 0.30$	90.000	3.098	3.129	3.123	3.115
$\delta = 0.07$	100.000	6.985	7.051	7.035	7.026
	110.000	12.882	12.988	12.955	12.947
	120.000	20.650	20.779	20.717	20.713
$r = 0.07$	80.000	1.664	1.664	1.664	1.657
$\sigma = 0.30$	90.000	4.495	4.495	4.495	4.483
$\delta = 0.03$	100.000	9.251	9.251	9.251	9.237
	110.000	15.798	15.798	15.798	15.784
	120.000	23.706	23.706	23.706	23.695

Table 1: American call options with the expiry $T = 3$(year) and the strike price $E = 100$. We discretized in 400 intervals in time and 2000 intervals in space. (Step size 0.0003 is used.)

Asset Price	80	90	100	110
Number of Meshes (L, M)		**Option Value**		
$(50, 300)$	0.213	1.343	4.705	11.038
$(100, 500)$	0.214	1.361	4.741	11.067
$(150, 700)$	0.215	1.369	4.755	11.077
$(200, 1000)$	0.217	1.377	4.768	11.087
$(300, 1500)$	0.218	1.381	4.774	11.092
Binomial Method	0.219	1.386	4.783	11.098

Table 2: Values of an American call option with $T = 0.5$, $E = 100$, $r = 0.03$, $\sigma = 0.20$, $\delta = 0.07$ by different mesh sizes (The step size 0.001 is used.)

the experiments represented in Table 1, we used very small mesh sizes. Also, the convergence property of the gradient projection algorithm is tested by experiments changing the mesh sizes. The result provided in Table 2 indeed shows the scheme converges as the number of meshes increases.

The performance of a popular method for the valuation of American options, the projected successive over-relaxation (PSOR) (see [21]), is compared with that of the gradient projection method. The computation result shown in Table 3 says that the gradient projection method is a competitive method in terms of accuracy and speed. Note that bigger step sizes are possible for more meshes in the gradient projection method; the bigger step size enables the algorithm to achieve the optimum faster. The over-relaxation parameter $\omega = 1.1$ is used for the PSOR methods.

The valuation of an American call option is presented graphically in Figures 1 and 2. All the computation in this paper was performed by MATLAB 7.0 at a generic desktop computer with dual Intel Xeon processors and 2GB RAM.

So far, we have studied and test the validity of our numerical method by comparing our results to results from other methods. In the Figure 3, the optimal timinig to invest and

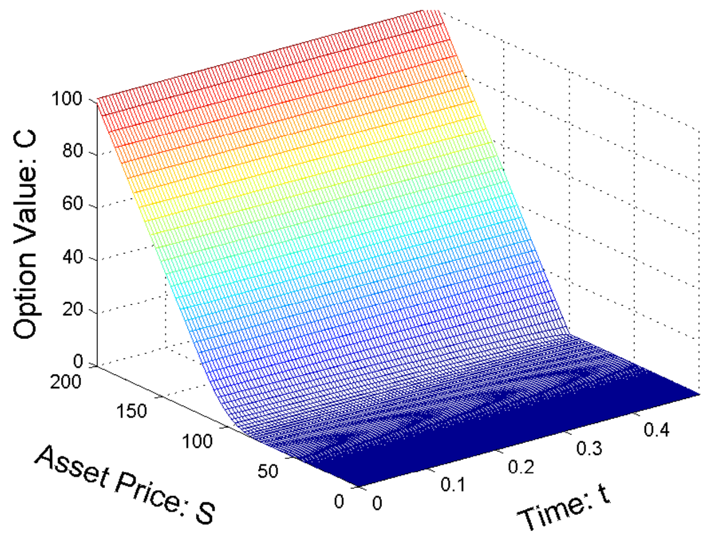

Figure 1: The result for an American call option when $E = 100$, $r = 0.03$, $\sigma = 0.2$, and $\delta = 0.07$.

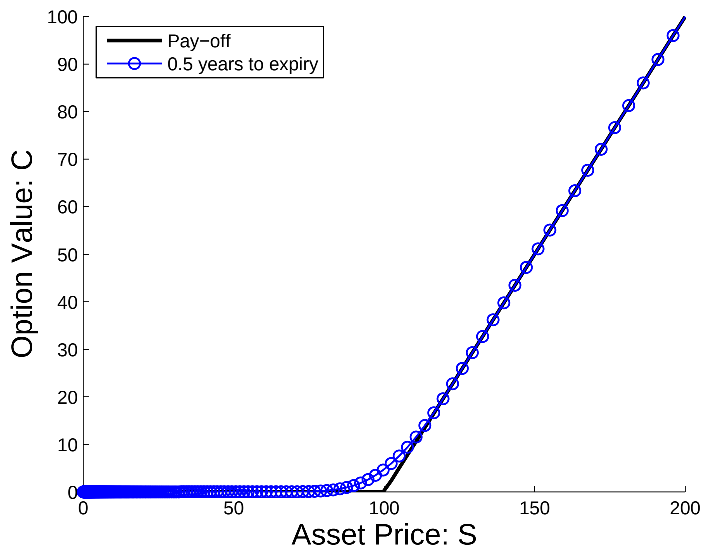

Figure 2: The result for an American call option when $T = 0.5$, $E = 100$, $r = 0.03$, $\sigma = 0.2$, and $\delta = 0.07$.

Asset Price	Binomial Method	$(L,M)=(20,100)$ Grad Proj	PSOR	$(L,M)=(50,300)$ Grad Proj	PSOR
80.000	0.219	0.2476	0.2476	0.2134	0.2134
90.000	1.386	1.2855	1.2855	1.3437	1.3437
100.000	4.783	4.5515	4.5515	4.7068	4.7068
110.000	11.098	10.9719	10.9719	11.0389	11.0389
120.000	20.000	20.0382	20.0382	19.9730	19.9730
	Cal Time	0.047	0.063	0.187	0.282
	Step Size	0.03		0.01	

Asset Price	Binomial Method	$(L,M)=(100,500)$ Grad Proj	PSOR	$(L,M)=(200,1000)$ Grad Proj	PSOR
80.000	0.219	0.2141	0.2142	0.2171	0.2171
90.000	1.386	1.3613	1.3616	1.3773	1.3774
100.000	4.783	4.7415	4.7420	4.7686	4.7687
110.000	11.098	11.0669	11.0672	11.0875	11.0876
120.000	20.000	19.9957	19.9957	19.9975	19.9975
	Cal Time	0.485	0.797	3.36	4.375
	Step Size	0.005		0.002	

Table 3: A Comparison between Gradient Projection Method and PSOR. $T = 0.5$, $E = 100$, $r = 0.03$, $\sigma = 0.20$, and $\delta = 0.07$. (Binomial method with 15,000 steps is used to compare the values.)

r	σ	δ	E	T_1	T_2	T_3	T_4
0.07	0.2	0.06	100	0.5	5	10	100

Table 4: Parameters

the values of investments for different time horizon were provided. The numerical examples were conducted based on the parameters in Table 4. And optimal timings corresponding to each expiration date which were obtained from the rule in the Section 4.2 is shown in Table. The results show that valuation of investment increases when the expiration date increases. Also, the investment value with a large expiration date tends to be close to the value with infinite expiration date. The optimal timing for infinite expiration date was obtained from a closed form solution, which will be discussed later.

6 Optimal Timing to Invest for Infinite Time Horizon Real Option Problems

If the expiration time of the option is infinite for the optimal investment timing problem, we can take an advantage of time independence in optimal solutions. And as shown in many literatures like Pindyck [17] and Weeds [19], a special form of optimal timing problems for infinite time horizon has a closed form solution for the investment decision. Similar to finite time horizon problem, $C(S,t)$ should satisfy:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta) S \frac{\partial C}{\partial S} - rC = 0$$

Herein, note that the time derivative term

Expiration Date(T)	0.5	5	10	100	∞
Optimal Investment timing (S^*)	141	165	183	183	187.9

Table 5: Change of Optimal Investment Timing

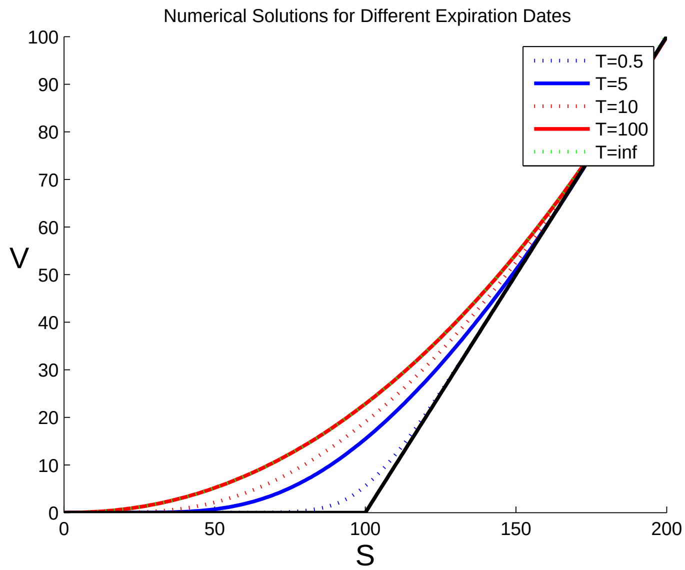

Figure 3: Numerical solutions for the different expiration date

$$\frac{\partial C}{\partial t} = 0$$

when $T = \infty$. Since C is not dependent on time t any more, we can write $C(S)$.

In addition, $C(S)$ need to satisfy the following boundary conditions:

$$C(0) = 0$$
$$C(S^*) = S^* - E$$
$$\frac{\partial C}{\partial S} = 1$$

The closed form solution are as follows:

$$C(S) = \begin{cases} aS^\beta & \text{for } S \le S^* \\ S - E & \text{for } S \le S^* \end{cases}$$

where

$$\beta = 1/2 - (r - \delta)/\sigma^2 + \left\{ \left((r - \delta)/\sigma^2 - 1/2 \right)^2 + 2r/\sigma^2 \right\}^{1/2}$$
$$S^* = \frac{\beta}{1 - \beta} E$$
$$a = (S^* - E)\,(S^*)^\beta$$

To derive LCP for infinite time horizon investment problem, let

$$\mathcal{L}_{BS}(C(S) = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta)S\frac{\partial C}{\partial S} - rC$$

We can considered $\mathcal{L}_{BS}(C(S) \leq 0$ because of no arbitrage assumption. We have the following two statements:

$$\text{If } C(S) > \max(S-E,0), \text{ then } \mathcal{L}_{BS}(C(S) = 0. \tag{16}$$

$$\text{If } C(S) = S-E = \max(S-E,0), \text{ then } \mathcal{L}_{BS}(C(S) < 0 \text{ since } \frac{\partial C}{\partial S} = 1 \tag{17}$$

Therefore, from these two inequality conditions, our optimal timing problem is formulated as a linear complementarity problem (LCP):

$$\begin{aligned} \mathcal{L}_{BS}(C(S)) \cdot [C(S) - \Phi(S)] &= 0 \\ C(S) - \Phi(S) &\geq 0 \\ -\mathcal{L}_{BS}(C(S)) &\geq 0 \end{aligned}$$

with boundary conditions

$$\begin{aligned} C\,(S) &\geq \Phi(S) \\ C\,(0) &= 0 \\ \lim_{S\to\infty} C\,(S) &\to \infty \end{aligned}$$

where we defined the Black-Scholes operator as

$$\mathcal{L}_{BS} \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-\delta)S\frac{\partial}{\partial S} - r$$

and the pay-off function as

$$\Phi(S) = \max(S-E,0)$$

Now we state that the solution to the LCP is indeed a solution to the real option problem of interest.

Lemma 8 *Solution of the LCP under assumption $r > \delta > 0$ always has the optimal threshold. The LCP is equivalent to the option to invest problem.*

Proof. In the LCP equation (16), the optimal timing is the expiration date which implies infinite time. However, if the condition $r > \delta > 0$ is satisfied, we know LCP can have a solution only from (16) by Wilmott et al. [21]. And we know that the exercise occurs immediately before the expiration date if the solution is obtained from (16). Therefore, the optimal threshold always exists within the finite time horizon. Since we know that the investment real option problem is the same as (16), our problem is equivalent to LCP. ■

We now ready to develop an extremal problem.

6.1 Variational Inequality and Extremal Problem

With the transformation introduced in Section 2.2, we have the following variational inequality problem for real option problems with infinite time horizon: to find $u \in \Omega$ for each time instant $\tau \in [0, T]$ such that

$$\int_{-\infty}^{\infty} \Psi(u)\,[v-u]\,dy \geq 0 \quad \forall v \in \Omega, \tag{18}$$

where the domain set is

$$\begin{aligned} \Omega = \{v : v - \phi \geq 0, v(y,0) = \phi(y), \\ v(-\infty,\tau) = \phi(-\infty), v(\infty,\tau) = \phi(\infty)\}, \end{aligned} \tag{19}$$

the principal operator is

$$\Psi = -\frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2} - \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial y} + r,$$

and the payoff function is

$$\phi(y) = \max(e^y - E, 0),$$

With the similar analysis as in Section 3, we have an equivalent extremal problem:

$$\min J\left(u\right) = \int_{-\infty}^{\infty}\int_{0}^{u} \Psi\left(v\right) dvdy \quad \text{s.t.} \quad u \in \Omega \tag{20}$$

where Ω is defined as (19). Also, we can show that the derivative of $J\left(u\right)$ is

$$\frac{\partial J(u)}{\partial u} = \Psi(u)$$

Note that for the infinite time horizon problems, the option value $u\left(\cdot\right)$ is not time-dependent but only depedent on the transformed asset price y.

6.2 A Numerical Method for Solving Infinite Time Horizon Problems

As the valuation function $u\left(\cdot\right)$ is not time-dependent, the finite difference approximation leads us to a one-dimensional problem, that is, we only discretize the y-axis. When we discretize the y-axis in M number of meshes limiting the domain of space y by an interval $[y_L, y_U]$ so that

$$\begin{aligned} y_i &= y_L + i\delta y, \qquad i = 0, ..., M \\ \delta y &= \frac{y_U - y_L}{M} \end{aligned}$$

Then the corresponding variational inequality is approximated to

$$\sum_{i=0}^{M} \Psi(u_i)\,[v_i - u_i] \geq 0 \quad \forall v \in \Omega \tag{21}$$

r	σ	δ	E
0.07	0.2	0.06	100

Table 6: A Comparison between Gradient Projection Method and PSOR. $T = 0.5$, $E = 100$, $r = 0.03$, $\sigma = 0.20$, and $\delta = 0.07$. (Binomial method with 15,000 steps is used to compare the values.)

where $u_i = u(y_i)$.

To approximate the parabolic operator $\Psi(\cdot)$, we consider finite difference approximations which leads

$$\Psi(u_i) \approx -\frac{1}{2}\sigma^2 \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{(\delta y)^2} \right) - \left(r - \delta - \frac{1}{2}\sigma^2 \right) \left(\frac{u_{i+1} - u_{i-1}}{2\delta y} \right) + r u_i \quad (22)$$

Then the gradient projection algorithm becomes similar to the numerical scheme introduced in Section 4, but at this time with the gradieint information (22).

7 Numerical Example

We have tested the gradient projection method proposed in the previous section for a real option problems with infinite time horizon for the parameters in Table 6. The numerical results produced are plotted in Figure 4, in which the closed-form solution is plotted in a solid line while the numerical solution is plotted in a dotted line.

The optimal timing to invest is obtained by the method in Section 4.2 as following:

$$S^*_{\text{numerical}} = 187.0000$$
$$S^*_{\text{closed-form}} = 187.9153$$

and the size of error is

$$\text{error} = |V_{\text{numerical}} - V_{\text{closed-form}}| = 0.7544$$

where the subscripts 'numerical' and 'closed-form' denote the values obtained by the numerical method and the closed-form solution, respectively.

8 Conclusion

In this paper, we constructed an extremal problem equivalent to the variational inequality formulation and discuss the gradient projection method for the extremal problem. To this end, we first studied the linear complimentarily problem form for real options, and basic algebraic manipulations enabled us to have the variational inequality formulations. We used some results of functional analysis such as G-differentiability and Riesz's Representation theorem to derive an extremal problem whose necessary condition coincides with the variational inequality formulation of American options.

Among infinite-dimensional optimization problems, the extremal problem we investigated has a few distinctive properties: (1) the domain set of the decision variable is defined

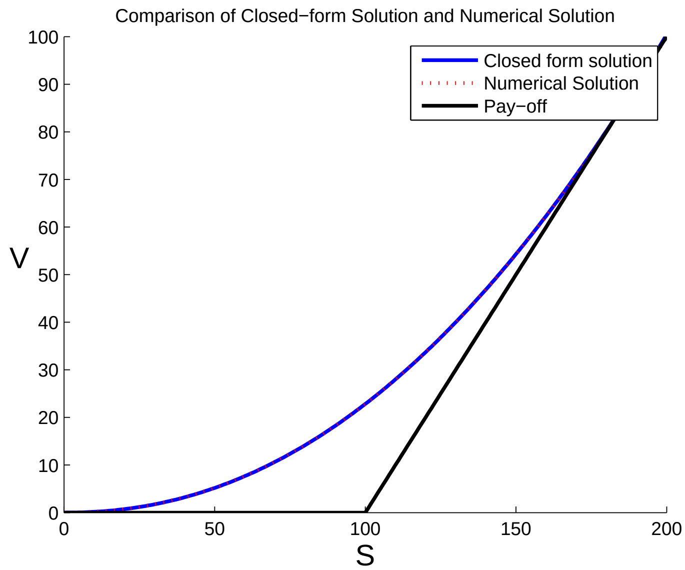

Figure 4: Comparison of Closed-form Solution and Numerical Solution for a Infinite Time Horizon Real Option Problem

by boundary conditions and an inequality, which is called an obstacle in traditional engineering problems, (2) the objective functional involves parabolic partial derivatives, and (3) the evaluation of the objective needs an integration from $-\infty$ to $+\infty$. These properties make a numerical approach to the solution difficult. We overcame the difficulties by iterative projections onto the domain, the Crank-Nicolson finite-difference approximations, and a finite length sub-interval approximation, respectively. We discovered that, when compared with binomial methods and projective successive over relaxation methods, the proposed gradient projection method gives fast and accurate solutions for several different American call options.

Since we approach the problem by optimizations, our method to the valuation of real options has an advantage: we can still value real options when different constraints to the value of options and/or the asset price are added. A straightforward method to the problem, then, is probably that we discretize and solve by nonlinear programming. These developments of solution algorithms are left for the future research works.

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